Research Article

Giovanni E. Comi and Kevin R. Payne*

On locally essentially bounded divergence measure fields and sets of locally finite perimeter

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Abstract: Chen, Torres and Ziemer ([9], 2009) proved the validity of generalized Gauss–Green formulas and obtained the existence of interior and exterior normal traces for essentially bounded divergence measure fields on sets of finite perimeter using an approximation theory through sets with a smooth boundary. However, it is known that the proof of a crucial approximation lemma contained a gap. Taking inspiration from a previous work of Chen and Torres ([7], 2005) and exploiting ideas of Vol'pert ([29], 1985) for essentially bounded fields with components of bounded variation, we present here a direct proof of generalized Gauss–Green formulas for essentially bounded divergence measure fields on sets of finite perimeter which includes the existence and essential boundedness of the normal traces. Our approach appears to be simpler since it does not require any special approximation theory for the domains and it relies only on the Leibniz rule for divergence measure fields. This freedom allows one to localize the constructions and to derive more general statements in a natural way.

Keywords: Divergence measure fields, sets of finite perimeter, generalized Gauss–Green theorems, normal traces, extension and gluing theorems

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1 Introduction

The Gauss–Green formula, or divergence theorem, plays a ubiquitous role in mathematical analysis, mathematical physics, and continuum physics by giving tools for establishing energy identities and energy inequalities for PDEs, for deriving the governing PDEs from basic physical principles and for rigorously justifying balance laws or conservation laws for classes of subbodies of a given body. Of particular importance is the search for extending the validity of such formulas to vector fields of lower regularity and for more general classes of subdomains. The literature is justifiably rich with such extensions, and below we will give a brief summary of some of the major developments which are most closely related to the present work. For a more complete review, see the monograph of Dafermos [11] and the extensive bibliography therein.

We are principally motivated by the paper of Chen–Torres–Ziemer [9] that examines the validity of the divergence theorem for *essentially bounded divergence measure fields* F on an open set $\Omega \subset \mathbb{R}^n$ and for subdomains $E \subset\subset \Omega$ of *finite perimeter in* Ω. Such vector fields are those $F \in L^{\infty}(\Omega; \mathbb{R}^n)$ whose distributional

Giovanni E. Comi, Classe di Scienze Matematiche e Naturali, Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy, e-mail: giovanni.comi@sns.it
***Corresponding author: Kevin R. Payne,** Dipartimento di Matematica "F. Enriques", Università di Milano, Via C. Saldini 50, 20133 Milano, Italy, e-mail: kevin.payne@unimi.it. http://orcid.org/0000-0002-7486-8940

divergence is a real finite Radon measure on Ω and such sets have characteristic functions χ_E which are of *bounded variation*; that is, they are L^1 and have distributional gradients which are $\mathbb{R}^n$-valued Radon measures on Ω. In this very general setting, the authors are able to incorporate *shock waves* in the form of jump surfaces, which are subsets of the boundary of a set E of finite perimeter on which the measure div F can concentrate and for which suitable notions of interior and exterior normal traces of F may not coincide.

In [9], in order to prove the Gauss–Green formula and to extract interior and exterior normal traces in the context specified above, the authors make use of an approximation theory for sets E of finite perimeter in $\mathbb{R}^n$ in terms of a family of smooth subsets which is well calibrated to any fixed Radon measure μ that is absolutely continuous with respect to the Hausdorff measure $\mathscr{H}^{n-1}$ (see [9, Theorem 4.10]). They first prove the result for sets with smooth boundary and then pass to a limit by exploiting their approximation theorem and a result of Šilhavý [25] which shows that if F is an essentially bounded divergence measure field, then the *total variation measure* $\mu = |\text{div}\, F|$ is absolutely continuous with respect to $\mathscr{H}^{n-1}$.

The principal aim of this paper is to show that this approximation step is not needed; that is, one can obtain the main result (see [9, Theorem 5.2]) directly by following the lines of Vol'pert's proof for essentially bounded BV vector fields and sets of finite perimeter (see [28] and [29]). To do so, one combines the aforementioned absolute continuity result of [25] with the Leibniz formula of Chen–Frid [5] (for the product of an essentially bounded function of bounded variation and an essentially bounded divergence measure field) and performs some elementary calculations of geometric measure theory. One might also note that in the aforementioned approximation result of [9], there was a known gap in the proof, which motivated our alternative method in the first place and has however been removed in the recent paper [10] by the first author and Torres. On the other hand, one should note that the approximation result is of independent interest and shows that for any set of finite perimeter E there exist sequences of smooth sets $E_{k,i}$ and $E_{k,e}$ converging to it from the interior and from the exterior in a measure theoretic sense (see [10, Theorem 3.1, Theorem 4.1 and Remark 4.1]). For instance, these sequences have been used by Chen–Torres in [8, proof of Lemma 4.3] and by Chen–Torres–Ziemer [9] for showing that the integrals of the generalized normal traces are indeed the limits of the integrals of the classical normal traces over the smooth sets which approximate E. A more detailed discussion of this point is given in Remark 3.11.

The advantage of our approach is its relative simplicity, since no approximation step is needed and no separate proof for smooth subdomains is required. Moreover, our method of proof leads easily to other relevant consequences, such as integration by parts formulas which also hold for domains with locally finite perimeter not necessarily compactly contained in Ω, when the test functions are compactly supported, and representation formulas for the measure div F on the *reduced boundary* of E and for the divergence measure of the gluing and the extension of essentially bounded divergence measure fields.

In order to place the present work into context, we now give a brief summary of some of the major developments in the search for generalized Gauss–Green and related formulas for vector fields of low regularity and rich classes of subdomains. A classical version of the Gauss–Green formula can asserts that for Ω an open subset of $\mathbb{R}^n$, if $F \in C^1(\Omega; \mathbb{R}^n)$ and $E \subset\subset \Omega$ is open with orientable boundary ∂E of class C^1, then[1]

$$\int_E \text{div}\, F\, dx = -\int_{\partial E} F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{1.1}$$

where ν_E is the *interior unit normal to* ∂E and $dx = d\mathcal{L}^n$, where $\mathcal{L}^n = \mathscr{H}^n$ is the Lebesgue measure on $\mathbb{R}^n$. We notice that (1.1) can be reformulated to say that there is a signed *divergence measure* μ and a signed *flux measure* σ on Ω such that

$$\mu(E) = \sigma(\partial E), \tag{1.2}$$

where μ is absolutely continuous with respect to $\mathcal{L}^n$ with a continuous density div F and σ is supported on the topological boundary ∂E and has the representation formula $\sigma = -(F \cdot \nu_E)\mathscr{H}^{n-1} \llcorner \partial E$ in terms of the trace of the normal component of F. Generalizations of (1.1) will be sought in the sense (1.2), where one searches for the precise meaning of μ and σ and their possible representations.

1 Here and throughout, we will write such formulas with respect to the interior normals.

A first important relaxation is found in the work of De Giorgi [12] and Federer [16] and involves Lipschitz vector fields F and $E \subset\subset \Omega$ of finite perimeter in Ω. In this setting, one has (1.1) if one replaces the topological boundary ∂E with the *reduced boundary* $\partial^* E$, which is contained in the support of $|D\chi_E|$, and interprets ν_E as the *measure theoretic interior normal*, which is well defined on $\partial^* E$. These fundamental notions of De Giorgi are recalled in Definition 2.11 and here we underline that their importance comes from the fact that an arbitrary set of finite perimeter can be very irregular; for example, its topological boundary can even have full Lebesgue measure $\mathcal{L}^n$. In this setting, the resulting Gauss–Green formula is

$$\int_E \operatorname{div} F\, dx = -\int_{\partial^* E} F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{1.3}$$

and it is worth mentioning that Federer's structure theory for sets of finite perimeter allows for inessential variants of (1.3), such as replacing $\partial^* E$ by the *measure theoretic boundary* $\partial^m E = \mathbb{R}^n \setminus (E^0 \cup E^1)$, where E^0 and E^1 are the *measure theoretic exterior* and *interior* respectively of E (as defined in (2.5) and (2.6)). The relevant structure theorem which justifies this claim is recalled in formulas (2.7)–(2.8) and we note that, for simplicity, we will work only with the notion of reduced boundary in the rest of this paper.

A second generalization is the aforementioned study of Vol'pert who extended the De Giorgi–Federer theory to include essentially bounded BV vector fields; that is, fields F whose components lie in $L^\infty(\Omega)$ and are of bounded variation on Ω. As mentioned, the scheme of Vol'pert's proof will be employed in the proof of our main result and hence a summary of the main steps is in order. The first ingredient is a product rule for essentially bounded BV functions; that is, if $u, v \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$, then $uv \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and

$$D(uv) = u^* Dv + v^* Du \quad \text{in the sense of } \mathbb{R}^n\text{-valued Radon measures on } \Omega, \tag{1.4}$$

where u^*, v^* are the *precise representatives of* u, v as defined in (2.9) and can be captured as the $\mathscr{H}^{n-1}$-a.e. limits of mollifications of u, v as recalled in (2.10). This step makes use of the important fact that for $u \in \mathrm{BV}(\Omega)$ one knows that the total variation measure $|Du|$ is absolutely continuous with respect to $\mathscr{H}^{n-1}$. The second ingredient involves showing that, roughly speaking, the distributional gradient of a compactly supported BV function has mean value zero, as happens for C^1_c-functions. This implies the Gauss–Green formula for compactly supported fields where there are no boundary terms. The last ingredient involves applying the product rule (1.4) to $u \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ and $v = \chi_E$, where $E \subset\subset \Omega$ is of finite perimeter in Ω. Performing some geometric measure theoretic manipulations on the resulting identity and using the compact support of $\chi_E u$ leads to a pair of generalized Gauss–Green formulas:

$$Du(E^1) = -\int_{\partial^* E} u_{\nu_E} \nu_E \, d\mathscr{H}^{n-1} \quad \text{and} \quad Du(E^1 \cup \partial^* E) = -\int_{\partial^* E} u_{-\nu_E} \nu_E \, d\mathscr{H}^{n-1}, \tag{1.5}$$

where $u_{\nu_E}(x), u_{-\nu_E}(x)$ are *interior, exterior traces* of u at $x \in \partial^* E$ which are $\mathscr{H}^{n-1}$-a.e. defined as the *approximate limits* of u restricted to the half-spaces $\Pi^{\pm}_{\nu_E}(x) := \{y \in \mathbb{R}^n : (y - x) \cdot (\pm\nu_E) \geq 0\}$. The precise meaning of this approximate limit is given in Remark 3.3. Applying (1.5) componentwise with $u = F_j \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ and $j = 1, \dots, n$ leads to

$$\operatorname{div} F(E^1) = -\int_{\partial^* E} F_{\nu_E} \cdot \nu_E \, d\mathscr{H}^{n-1} \quad \text{and} \quad \operatorname{divF}(E^1 \cup \partial^* E) = -\int_{\partial^* E} F_{-\nu_E} \cdot \nu_E \, d\mathscr{H}^{n-1}. \tag{1.6}$$

A final group of generalizations stems from the observation that a vector field F can have its distributional divergence be a Radon measure without having the distributional gradient of each component F_j of F be a Radon measure. Moreover, one can attempt to relax the requirement that F be essentially bounded. This motivates the introduction of the space $\mathcal{DM}^p(\Omega; \mathbb{R}^n)$ of *p-summable divergence measure fields* for $p \in [1, \infty]$, made up of those $F \in L^p(\Omega; \mathbb{R}^n)$ for which the distributional divergence $\operatorname{div} F$ is a finite real Radon measure on Ω. The case $p = \infty$ of essentially bounded divergence measure fields, and their local versions, will be the focus of our interest. Such fields were first introduced by Anzellotti [3] when $p = \infty$ in his study of pairings between measures and bounded functions. Amongst other things, this study led to the existence of $L^\infty(\partial\Omega)$ traces of the normal component of essentially bounded divergence measure fields on the boundary of open bounded sets Ω with Lipschitz boundary (see [3, Theorem 1.2]). Such traces are called *normal traces* in the literature. Later, such fields were studied by many authors having in mind various applications and resulting in new versions of the Gauss–Green formula. In particular, motivated by applications to the theory of systems of conservation laws with the Lax entropy condition, Chen and Frid proved generalized Gauss–Green formulas for divergence measure fields on open bounded sets with *Lipschitz deformable boundary* (see [5, Theorem 2.2] and [6, Theorem 3.1]). Moreover, Chen, Torres and Ziemer extended this result to the sets of finite perimeter in the case $p = \infty$ in [9, Theorem 5.2] and later Chen and Torres applied this theorem to the study of solutions of nonlinear hyperbolic systems of conservation laws ([8]). It is [9, Theorem 5.2] that we wish to reexamine in this paper as will be further specified below after recalling some additional related results.

As a means of comparison, some additional works concerning divergence measure fields should be mentioned. Degiovanni, Marzocchi and Musesti in [13] and later Schuricht in [24] sought to prove the existence of normal traces under weak regularity hypotheses in order to achieve a representation formula for Cauchy fluxes, contact interactions and forces in the context of the foundations of continuum physics. In particular, a justification of *Cauchy's stress theorem* under weak regularity assumptions is a main unifying ingredient in much of the divergence measure field literature, as is well explained in the introduction of [24]. While the resulting Gauss–Green formulas (and justifications of the stress theorem) obtained in [13] and [24] are valid for $\mathcal{DM}^p(\Omega; \mathbb{R}^n)$-fields for any $p \geq 1$, the subdomains E cannot be taken to be arbitrary sets of finite perimeter. Instead, E must be chosen to lie in a suitable subalgebra of sets which are related to the particular vector field F. On the other hand, Ziemer [30] established the Cauchy stress theorem with respect to subdomains of finite perimeter for divergence measure fields under the additional assumption that $\operatorname{div} F \in L^\infty(\Omega)$. Another important work along these lines is the study of Cauchy fluxes in Šilhavý [25], who sought to give a more complete description of generalized Gauss–Green formulas for $\mathcal{DM}^p(\Omega; \mathbb{R}^n)$-fields with respect to the values of $p \in [1, \infty]$ and concentration hypotheses on $\operatorname{div} F$. In particular, he gave sufficient conditions under which the interior and exterior normal traces can be seen as integrable functions with respect to the measure $\mathscr{H}^{n-1}$ on the reduced boundary of a set of finite perimeter. Such conditions are always satisfied in the case $p = \infty$ and we will show in Example 6.1 that this is indeed the only case in which this happens in general, by constructing a counterexample in $\mathcal{DM}^p$ for any $p \in [1, \infty)$. It is worth noting that Šilhavý also studied the properties of the so-called *extended divergence measure fields*, already introduced by Chen–Frid in [6], which are vector-valued Radon measure whose divergence is still a Radon measure. He showed absolute continuity results and Gauss–Green formulas in [26] and [27]. One should also mention the work of Ambrosio, Crippa and Maniglia [1] which aimed at extensions of the DiPerna–Lions theory for transport equations at low regularity. They studied a class of these vector fields induced by *functions of bounded deformation* and proved a Gauss–Green formula for essentially bounded divergence measure fields on open sets with C^1-boundary compactly contained in the domain. Finally, it might be noted that in their study of mean value properties of harmonic functions on metric spaces (X, d) supporting a doubling measure μ and a $(1, 1)$-Poincaré inequality, Marola, Miranda and Shanmugalingam [21] verified the validity of generalized Gauss–Green theorems on balls in metric spaces for 2-summable divergence measure fields.

We now return to the content of the present paper. As already mentioned, the main idea is to present a new proof of the Gauss–Green formula for essentially bounded divergence measure fields F on Ω for sets of finite perimeter $E \subset\subset \Omega$. Carefully studying the paper of Chen and Torres [7], we noticed that it was possible to work directly with E along the lines of Vol'pert's proof for essentially bounded BV-vector fields, which was sketched above. Hence we are able to avoid the need to approximate E from the interior by smooth domains. While the statement of the fundamental result (Theorem 3.2) is essentially the same as the main result in Theorem 5.2 of Chen, Torres and Ziemer [9], our proof is much simpler. Indeed, beyond known facts from geometric measure theory concerning sets of finite perimeter and functions of bounded variation, it relies only on the following three ingredients for essentially bounded divergence measure fields $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$:

(1) The absolute continuity property of the divergence of the field: $|\operatorname{div} F| \ll \mathscr{H}^{n-1}$.

(2) The Leibniz rule of [5]: if $g \in BV(\Omega) \cap L^\infty(\Omega)$, then $gF \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$ and

$$\operatorname{div}(gF) = g^* \operatorname{div} F + \overline{F \cdot Dg},$$

where g^* is the precise representative of g and $\overline{F \cdot Dg}$ is a Radon measure, which is the weak-star limit of a radially mollified sequence $F \cdot \nabla(g * \rho_\delta)$ and is absolutely continuous with respect to $|Dg|$.

(3) The divergence theorem in the case of compactly supported vector fields: if F has compact support in Ω, then $\operatorname{div} F(\Omega) = 0$.

The main result will state that if $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ and if $E \subset\subset \Omega$ is a set of finite perimeter in Ω, then there exist *interior and exterior normal traces of F on $\partial^* E$*; that is, $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty(\partial^* E; \mathscr{H}^{n-1})$ such that a pair of Gauss–Green formulas analogous to (1.6) hold:

$$\operatorname{div} F(E^1) = -2\overline{\chi_E F \cdot D\chi_E}(\partial^* E) = -\int_{\partial^* E} \mathcal{F}_i \cdot \nu_E \, d\mathscr{H}^{n-1}$$

and

$$\operatorname{div} F(E^1 \cup \partial^* E) = -2\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}(\partial^* E) = -\int_{\partial^* E} \mathcal{F}_e \cdot \nu_E \, d\mathscr{H}^{n-1},$$

where $\overline{\chi_E F \cdot D\chi_E}$ and $\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}$ are the weak star limits, respectively, of the sequences $\chi_E F \cdot \nabla(\chi_E * \rho_\delta)$ and $\chi_{\Omega\setminus E} F \cdot \nabla(\chi_E * \rho_\delta)$ as $\delta \to 0$, up to a subsequence. Moreover, one will have the following trace estimates:

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\partial^* E;\mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(E;\mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\partial^* E;\mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}.$$

We notice that this new proof also adjusts a dubious point in the proof of the Gauss–Green formula in [7]; indeed, [7, formula (44)], which states

$$\overline{\chi_E F \cdot D\chi_E} = \chi_E^* \overline{F \cdot D\chi_E},$$

is false in general for a vector field $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ and a set $E \subset\subset \Omega$ of finite perimeter in Ω (see also Remark 3.4). In addition, this method of proof yields immediately many relevant consequences, such as the representation formula for $\operatorname{div} F$ on the reduced boundary of sets of finite perimeter, integration by parts formulas and various results on gluing constructions which come from the ability to directly localize constructions as one does not need to pass through an approximation procedure.

We conclude with a brief summary of the contents of the present work. In Section 2, we give the necessary background and preliminary results on Radon measures, sets of finite perimeter and divergence measure fields, including the needed ingredients (1) and (2) listed above (see Corollary 2.16 and Theorem 2.18). In Section 3, after proving the main result on the Gauss–Green formulas in Theorem 3.2 for $\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$-fields, we derive some useful corollaries including a representation of the measure divF on the reduced boundary and a version for fields which are locally essentially bounded divergence measure fields $F \in \mathcal{DM}^\infty_{\text{loc}}(\Omega;\mathbb{R}^n)$ (see Corollaries 3.5 and 3.6). We also prove that, in the case of continuous fields F, the normal trace is the classical dot product in Theorem 3.7. In Section 4, we present various integration by parts formulas for $F \in \mathcal{DM}^\infty_{\text{loc}}(\Omega;\mathbb{R}^n)$ and locally Lipschitz functions φ on sets of locally finite perimeter E and discuss some applications including improved L^∞-estimates of the normal traces. We also discuss the determination of normal traces in Proposition 4.10. In Section 5, we present two gluing constructions for building $\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$-fields out of a pair of $\mathcal{DM}^\infty$-fields whose domains decompose Ω, with or without essential overlap. These results are similar to results presented in [7] and [9]. Ultimately, we will use these constructions also to obtain Gauss–Green and integration by parts formulas up to the boundary of open bounded sets with regular enough boundary in Corollary 5.5. Finally, in Section 6 we will make some concluding remarks concerning the role of $p = \infty$ and some additional comparisons with the literature including alternate representation formulas for the normal traces.

2 Notation and preliminary results

In this section, we wish to set the notations we will use and present the necessary preliminaries for the main results in the following sections. In particular, we will need some known facts from abstract measure theory, including weak convergence of Radon measures and elements of geometric measure theory including Hausdorff measures, capacity and elements of the Caccioppoli–De Giorgi–Federer theory of sets of finite perimeter. The notion of divergence measure fields will be recalled, together with some important preliminary results concerning the absolute continuity of $\operatorname{div} F$ with respect to $\mathscr{H}^{n-1}$ and the crucial Leibniz formula for products of essentially bounded functions of bounded variation and essentially bounded divergence measure fields. We will attempt to be brief while keeping the exposition relatively self-contained.

We begin with some notation. In the rest of the paper, Ω is an open subset of $\mathbb{R}^n$ and $\subset$ is equivalent to $\subseteq$. The symmetric difference of sets is denoted by $A \Delta B := (A \setminus B) \cup (B \setminus A)$. We denote by $E \subset\subset \Omega$ a set E whose closure, $\overline{E}$, is compact and contained in Ω, by E° the interior of the set E and by ∂E its topological boundary.

We denote by $\mathscr{L}^n$ and $\mathscr{H}^\alpha$ the Lebesgue and α-dimensional Hausdorff measures on $\mathbb{R}^n$, where $\alpha \geq 0$. Unless otherwise stated, a measurable set is an $\mathscr{L}^n$-measurable set. For any measurable set $E \subset \mathbb{R}^n$, we denote by $|E|$ the $\mathscr{L}^n$-measure of E, while, when applied to a function with values in $\mathbb{R}^m$, $|\cdot|$ is the euclidian norm. As usual, $B(x, r)$ is the open ball with center in x and radius $r > 0$ and $\omega_n = |B(0, 1)|$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$, where $\mathscr{H}^{n-1}(\mathbb{S}^{n-1}) = n\omega_n$. We will denote by $\mathcal{B}(\Omega)$ the Borel sigma algebra generated by the open subsets of $(\Omega, |\cdot|)$ which is a locally compact and separable metric space. We also use the standard notations $\mu \llcorner A$ for the restriction of a measure μ to the set A and $\mu \ll \nu$ to indicate that the measure μ is absolutely continuous with respect to the measure ν.

For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$ we denote by $C_c^k(\Omega; \mathbb{R}^m) := \{\varphi \in C^k(\Omega; \mathbb{R}^m) : \operatorname{supp}(\varphi) \subset\subset \Omega\}$ the space of C^k functions compactly supported in Ω which will be endowed with the sup norm

$$\|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)|.$$

We denote by $\mathrm{Lip}(\Omega)$, $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and $\mathrm{Lip}_c(\Omega)$ the spaces of Lipschitz, locally Lipschitz and Lipschitz functions with compact support in Ω, respectively.

2.1 Radon measures and weak-star convergence

The needed calculus for divergence measure fields operates in the context of real-signed and vector-valued Radon measures. Hence elements of this general measure theory are essential for the development. We have followed essentially the treatments of the monographs Ambrosio–Fusco–Pallara [2] and Evans–Gariepy [14], which contain the proofs of the results merely stated herein.

We begin with the notions of Radon measures and their total variation.

Definition 2.1. Let Ω be open in $\mathbb{R}^n$.

(a) A measure μ on Ω is called a *positive Radon measure* on Ω if μ is nonnegative, every $B \in \mathcal{B}(\Omega)$ is μ-measurable, and μ is finite on the compact subsets of Ω.

(b) A real-signed (or vector-valued) measure in Ω is called a *real-signed (or vector-valued) Radon measure* on Ω if it is defined on $\mathcal{B}(K)$ for any compact subset K of Ω and the total variation of μ is finite on every compact $K \subset \Omega$. This means that the total variation measure

$$|\mu|(B) := \sup \left\{ \sum_{k=0}^{+\infty} |\mu(B_k)| : B_k \text{ Borel sets pairwise disjoint, } B = \bigcup_{k=0}^{+\infty} B_k \right\},$$

defined on $\mathcal{B}(\Omega)$ is finite on the compact subsets of Ω.

(c) The space of real Radon measures on Ω is denoted by $\mathcal{M}_{\mathrm{loc}}(\Omega)$ and the space of $\mathbb{R}^m$-vector-valued Radon measures by $\mathcal{M}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$. In addition, if $|\mu|(\Omega) < \infty$, then μ is a *(real-signed or vector-valued) finite Radon measure* and we write $\mu \in \mathcal{M}(\Omega)$ or $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$, if it is vector-valued.

It is well known that any positive Radon measure is inner and outer regular; that is, for any $B \in \mathcal{B}(\Omega)$,

$$\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact}\} \quad \text{and} \quad \mu(B) = \inf\{\mu(U) : B \subset U,\ U \text{ open}\}. \tag{2.1}$$

In addition, each $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ determines a positive Radon measure, the *total variation measure* $|\mu|$, which is given by its values on open subsets $A \subset \Omega$ through the formula

$$|\mu|(A) := \sup\left\{ \int_\Omega \varphi \cdot d\mu : \varphi \in C_c(A; \mathbb{R}^m),\ \|\varphi\|_\infty \leq 1 \right\}.$$

Since μ is absolutely continuous with respect to $|\mu|$, the Radon–Nikodym theorem and elementary considerations give rise to the *polar decomposition of* μ; that is, there exists a unique $f \in L^1(\Omega, |\mu|; \mathbb{R}^m)$ with $|f(x)| = 1$ for $|\mu|$-a.e. $x \in \Omega$ such that

$$\mu = f|\mu|. \tag{2.2}$$

For these results, we refer to [2, Proposition 1.43, Proposition 1.47 and Corollary 1.29].

We now briefly discuss the notion of weak-star convergence and a compactness criterion for such measures. The Riesz representation theorem shows that the spaces $\mathcal{M}_{\rm loc}(\Omega; \mathbb{R}^m)$ and $\mathcal{M}(\Omega; \mathbb{R}^m)$ can be identified with the duals of $C_c(\Omega; \mathbb{R}^m)$ and $C_0(\Omega; \mathbb{R}^m)$, respectively, where $C_0(\Omega; \mathbb{R}^m)$ is the completion of $C_c(\Omega; \mathbb{R}^m)$ with respect to the sup norm; that is, the space of continuous functions φ on Ω satisfying the property: for any $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|\varphi(x)| < \varepsilon$ for each $x \in \Omega \setminus K$.

Definition 2.2. Given a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega)$, one says that μ_k *weak-star converges to* $\mu \in \mathcal{M}(\Omega)$ if

$$\lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \quad \text{for all } \varphi \in C_0(\Omega; \mathbb{R}^m).$$

If $\{\mu_k\}$ and μ are in $\mathcal{M}_{\rm loc}(\Omega)$, one says that μ_k *locally weak-star converges to* μ if

$$\lim_{k\to+\infty} \int_\Omega \varphi \cdot d\mu_k = \int_\Omega \varphi \cdot d\mu \quad \text{for all } \varphi \in C_c(\Omega; \mathbb{R}^m).$$

Necessary conditions for the weak-star convergence $\mu_k \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega; \mathbb{R}^m)$ are

$$\limsup_{k\to+\infty} |\mu_k|(\Omega) < \infty \quad \text{and} \quad |\mu|(\Omega) \le \liminf_{k\to+\infty} |\mu_k|(\Omega),$$

which follow from the definition of dual norm and the Uniform Boundedness Principle.

More importantly, one has the following weak compactness criterion in $\mathcal{M}(\Omega; \mathbb{R}^m)$:

$$\sup\{|\mu_k|(\Omega) : k \in \mathbb{N}\} < +\infty \implies \text{there exists a weak-star converging subsequence of } \{\mu_k\};$$

see [2, Theorem 1.59].

Remark 2.3. As is well known, weak-star convergence of finite Radon measures is implied by the local weak-star convergence under the condition that $\sup |\mu_k|(\Omega) = C < \infty$. We observe that this condition implies $|\mu|(\Omega) \le C$. Therefore, in what follows, we will always write $\mu_k \overset{*}{\rightharpoonup} \mu$ to denote local weak-star convergence, and, in the case of finite Radon measures, we will also check the condition $\sup |\mu_k|(\Omega) < \infty$.

We quote now a useful result concerning weak-star convergence which will play a key role in the L^∞-estimates of the normal traces of the Gauss–Green formula of Theorem 3.2.

Lemma 2.4. *Let $\mu_k, \mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ and let ν be a positive Radon measure on Ω such that $\mu_k \overset{*}{\rightharpoonup} \mu$ and $|\mu_k| \overset{*}{\rightharpoonup} \nu$. Then the following statements hold:*

(1) *$|\mu| \le \nu$ and for each μ-measurable set $E \subset\subset \Omega$ satisfying $\nu(\partial E) = 0$ one has*

$$\mu(E) = \lim_{k\to+\infty} \mu_k(E) \quad \text{and} \quad \nu(E) = \lim_{k\to+\infty} |\mu_k|(E).$$

(2) *Given any $x \in \Omega$ and $R = R_x > 0$ such that $B(x, R) \subset\subset \Omega$, then for $\mathscr{L}^1$-a.e. $r \in (0, R)$ we have*

$$\nu(\partial B(x, r)) = 0.$$

Hence, $\mu_k(B(x, r)) \to \mu(B(x, r))$ and $|\mu_k|(B(x, r)) \to \nu(B(x, r))$ for such values of r.

Proof. For point (1) we refer to [2, Proposition 1.62]. For completeness, we recall briefly the proof of point (2), which follows from the fact that there are limitations on how much a Radon measure can concentrate. For an interval I, if $\{A_t\}_{t\in I}$ is a family of ν-measurable relatively compact sets in Ω such that the sets ∂A_t are pairwise disjoint, then there exists a countable set $\mathcal{N}$ such that $\nu(\partial A_t) = 0$ for all $t \in I \setminus \mathcal{N}$. Indeed, since ν is finite on bounded sets and additive, the set $\{t \in I : \nu(\partial A_t) > \varepsilon\}$ is finite for any $\varepsilon > 0$. This implies that the set $\{t \in I : \nu(\partial A_t) > 0\}$ is at most countable (see also [2, observation at the end of Section 1.4]). By applying this argument to the family $\{B(x, r)\}_{r\in(0,R)}$, one has that $\nu(\partial B(x, r)) = 0$ for $\mathscr{L}^1$-a.e. $r \in (0, R)$. Hence, by using point (1), one concludes the proof of (2). □

2.2 Relative capacity and relations to Hausdorff measures

As is well known, the notion of capacity is very useful in the study of the fine properties of Sobolev functions and for Sobolev-type inequalities for functions of bounded variation. Herein the notion will play a key role in the proof of the absolute continuity of divergence measures with respect to Hausdorff measures (Corollary 2.16 which depends on Theorem 2.15). The brief exposition here borrows from the monographs of Maz'ya [22], Heinonen–Kilpeläinen–Martio [20] and Evans–Gariepy [14].

Definition 2.5. For $1 \le p \le n$ and a compact subset K of the open set Ω in $\mathbb{R}^n$, we define the *p-capacity* of K relative to Ω as

$$\operatorname{cap}_p(K,\Omega) := \inf\Big\{\int_\Omega |\nabla\varphi|^p\,dx : \varphi \in C_c^\infty(\Omega),\ \varphi \ge 1 \text{ on } K\Big\}.$$

If $U \subset \Omega$ is open, we set

$$\operatorname{cap}_p(U,\Omega) := \sup\{\operatorname{cap}_p(K,\Omega) : K \subset U \text{ compact}\}$$

and, for an arbitrary set $A \subset \Omega$,

$$\operatorname{cap}_p(A,\Omega) := \inf\{\operatorname{cap}_p(U,\Omega) : A \subset U \subset \Omega,\ U \text{ open}\}.$$

If $\Omega = \mathbb{R}^n$, we write $\operatorname{cap}_p(A,\mathbb{R}^n) = \operatorname{cap}_p(A)$ for any set A.

For any compact subset K of Ω, Definition 2.5 is equivalent to

$$\operatorname{cap}_p(K,\Omega) := \inf\Big\{\int_\Omega |\nabla\varphi|^p\,dx : \varphi \in C_c^\infty(\Omega),\ 0 \le \varphi \le 1,\ \{\varphi = 1\}^\circ \supset K\Big\},$$

which follows from an approximation argument that one finds in [22, Section 2.2.1, point (ii)]. We shall use the following well-known monotonicity properties of the capacity:
(1) If Ω_1, Ω_2, with $\Omega_1 \subset \Omega_2$, are open and $A \subset \Omega_1$, then $\operatorname{cap}_p(A,\Omega_2) \le \operatorname{cap}_p(A,\Omega_1)$. In particular, if $\Omega_2 = \mathbb{R}^n$, then $\operatorname{cap}_p(A) \le \operatorname{cap}_p(A,\Omega)$ for any open set Ω and any set $A \subset \Omega$.
(2) If $A_1 \subset A_2 \subset \Omega$, then
$$\operatorname{cap}_p(A_1,\Omega) \le \operatorname{cap}_p(A_2,\Omega). \tag{2.3}$$

We recall a classical result which shows the relations between the p-capacity and the $(n-p)$-Hausdorff measure.

Theorem 2.6. *The following statements hold:*
(a) *Let $K \subset \Omega$ be a compact set. If $1 < p < n$, then $\mathscr{H}^{n-p}(K) < \infty$ implies $\operatorname{cap}_p(K,\Omega) = 0$, while, if $p = 1$, one has $\mathscr{H}^{n-1}(K) = 0$ if and only if $\operatorname{cap}_1(K,\Omega) = 0$.*
(b) *In addition, if $\Omega = \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, then:*
(1) *If $1 < p < n$ and $\operatorname{cap}_p(A) = 0$, then $\mathscr{H}^s(A) = 0$ for $s > n-p$.*
(2) *$\operatorname{cap}_1(A) = 0$ if and only if $\mathscr{H}^{n-1}(A) = 0$.*

When $1 < p < n$, part (a) of the theorem is the content of [20, Theorem 2.27] while part (b) is the content of [14, Theorem 4 of Section 4.7.2]. When $p = 1$, the theorem follows from [19, Theorems 4.4 and 5.1]. More precisely, they show that cap_1 is comparable to a BV notion of capacity and that this BV capacity and $\mathscr{H}^{n-1}$ vanish on the same sets. See also [14, Theorem 3 of Section 5.6.3] for a proof of part (b) when $p = 1$ by using isoperimetric inequalities.

Next, we state a useful result concerning the compact sets $K \subset \Omega$ for which $\operatorname{cap}_p(K,\Omega) = 0$. Such a result can be also seen as an easy consequence of [20, Lemma 2.9]; however, we will give an alternate proof which does not use the notion of sets of zero p-capacity and Choquet's capacitability theorem. Instead, we make use of the Gagliardo–Nirenberg–Sobolev inequality.

Lemma 2.7. *Let $1 \le p < n$ and let K be a compact set such that $\operatorname{cap}_p(K,\Omega) = 0$ for an open set $\Omega \supset K$. Then $\operatorname{cap}_p(K,\Omega') = 0$ for any bounded open set Ω' satisfying $K \subset \Omega' \subset\subset \Omega$.*

Proof. Let Ω' be such that $K \subset \Omega' \subset\subset \Omega$. We take $\varphi \in C_c^\infty(\Omega)$, $0 \le \varphi \le 1$, $\{\varphi = 1\}^\circ \supset K$ and $\psi \in C_c^\infty(\Omega')$, $0 \le \psi \le 1$, $\{\psi = 1\}^\circ \supset K$, then $\varphi\psi \in C_c^\infty(\Omega')$, $0 \le \varphi\psi \le 1$, $\{\varphi\psi = 1\}^\circ \supset K$. Thus

$$\begin{aligned} \operatorname{cap}_p(K, \Omega') &\le \int_{\Omega'} |\nabla(\varphi\psi)|^p \, dx \le 2^p \Big(\int_{\Omega} |\nabla\varphi|^p \, dx + \|\nabla\psi\|_\infty^p \int_{\Omega'} |\varphi|^p \, dx \Big) \\ &\le 2^p \Big(\int_{\Omega} |\nabla\varphi|^p \, dx + \|\nabla\psi\|_\infty^p |\Omega'|^{\frac{p}{n}} \|\varphi\|_{L^{p^*}(\Omega)}^p \Big) \le C(\nabla\psi, \Omega', p) \int_{\Omega} |\nabla\varphi|^p \, dx, \end{aligned}$$

Taking the inf over φ gives the result, since $\operatorname{cap}_p(K, \Omega) = 0$. □

We now state the technical lemma which will be used to show the absolute continuity of the distributional divergence of divergence measure fields. A similar result was shown in [23], in the proof of Theorem 2.8.

Lemma 2.8. *Let $1 \le p < n$ and let K be a compact subset of Ω. If $\operatorname{cap}_p(K, \Omega) = 0$, then there exists a sequence of functions $\varphi_j \in C_c^\infty(\Omega)$ such that*
(i) $0 \le \varphi_j \le 1$ *and* $\varphi_j = 1$ *on* K,
(ii) $\|\nabla\varphi_j\|_{L^p(\Omega;\mathbb{R}^n)} \to 0$,
(iii) *for each j, $\operatorname{supp}(\varphi_j)$ is contained in an open set $U_j \subset\subset \Omega$ such that $\{U_j\}$ is a decreasing sequence and $\bigcap_{j=1}^\infty U_j = K$, which implies $\varphi_j(x) \to 0$ for all $x \in \Omega \setminus K$.*

Proof. Since K is a compact set and $\operatorname{cap}_p(K, \Omega) = 0$, it follows from Lemma 2.7 that $\operatorname{cap}_p(K, U) = 0$ for any open set U such that $K \subset U \subset\subset \Omega$. By selecting a decreasing sequence of open sets U_j such that $U_j \subset\subset \Omega$ and $\bigcap_{j=1}^{+\infty} U_j = K$, one concludes that $\operatorname{cap}_p(K, U_j) = 0$ for any j. Therefore, one can find $\varphi_j \in C_c^\infty(U_j)$ such that $0 \le \varphi_j \le 1$, $\varphi_j = 1$ on a neighborhood of K and $\|\nabla\varphi_j\|_{L^p(\Omega;\mathbb{R}^n)} \to 0$. Finally, if $x \notin K$, then $x \notin U_j$ for any $j \ge j_0$, for some j_0, which easily implies $\varphi_j(x) \to 0$. □

One could extend Lemma 2.8 to the borderline case $p = n$ (with a similar proof) if one makes use of the full $W^{1,p}$-Sobolev capacity, which employs the full functional $\int_\Omega [|\nabla\varphi|^p + |\varphi|^p] \, dx$ in the definition of relative p-capacity (Definition 2.5). Alternatively, one can apply the notion of p-capacity and Choquet's capacitability theorem, as done in [20].

2.3 Functions of bounded variation and sets of finite perimeter

We recall now a few basic definitions and results in the theory of functions of bounded variation and sets of finite perimeter[2], which give essential ingredients in the framework for generalized Gauss–Green theorems. In particular, we will make use of elements in the structure theory of sets of finite perimeter as developed by De Giorgi [12] and Federer [16] (see also the manuscript of Federer [17]). We follow mainly the treatment of the monographs [2] and [14] and additional facts will be recalled later, when they are needed.

Definition 2.9. Let $\Omega \subset \mathbb{R}^n$ be open.
(a) A function $u \in L^1(\Omega)$ is said to be of *bounded variation in* Ω if the distributional gradient Du is a finite $\mathbb{R}^n$-vector-valued Radon measure on Ω and the space of all such functions will be denoted by $\mathrm{BV}(\Omega)$. One says that u is of *locally of bounded variation in* Ω if for every open set $W \subset\subset \Omega$ one has $u_{|W} \in \mathrm{BV}(W)$; the space of all such functions will be denoted by $\mathrm{BV}_{\mathrm{loc}}(\Omega)$.
(b) A measurable set $E \subset \Omega$ is said to be a *set of finite perimeter in* Ω if $\chi_E \in \mathrm{BV}(\Omega)$ and said to have *locally finite perimeter* in Ω if $\chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$.

Consequently, $D\chi_E$ is an $\mathbb{R}^n$-vector-valued Radon measure on Ω whose total variation is $|D\chi_E|$ and, by the polar decomposition of measures (2.2), one can write $D\chi_E = \nu_E |D\chi_E|$, where ν_E is a $|D\chi_E|$-measurable function such that $|\nu_E(x)| = 1$ for $|D\chi_E|$-a.e. $x \in \Omega$.

2 Such sets are also known as Caccioppoli sets.

Remark 2.10. Important examples of sets of finite perimeter in Ω are open bounded sets $U \subset\subset \Omega$ such that $\mathscr{H}^{n-1}(\partial U) < \infty$ or ∂U is Lipschitz. In this second case, it is possible to show that

$$|D\chi_U| = \mathscr{H}^{n-1} \llcorner \partial U, \tag{2.4}$$

as is known from the work of Federer (see [2, Proposition 3.62], for example).

While (2.4) says that $|D\chi_U|$ is concentrated on the topological boundary of a bounded Lipschitz domain U, this does not happen in general. Indeed, the topological boundary of a bounded set of finite perimeter E can be very irregular, including the possibility of having positive Lebesgue measure $\mathcal{L}^n$. On the other hand, De Giorgi [12] discovered a suitable subset of ∂E of finite $\mathscr{H}^{n-1}$-measure on which $|D\chi_E|$ is concentrated if E has finite perimeter in Ω.

Definition 2.11. Let E be a measurable subset of $\mathbb{R}^n$ and let Ω be the largest open subset for which E is of locally finite perimeter in Ω. The *reduced boundary* of E, denoted by $\partial^* E$, is defined as the set of all $x \in \operatorname{supp}(|D\chi_E|) \cap \Omega$ such that the limit

$$\nu_E(x) := \lim_{r\to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$$

exists in $\mathbb{R}^n$ and satisfies

$$|\nu_E(x)| = 1.$$

The function $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ is called the *measure theoretic unit interior normal* to E.

A precise justification for calling ν_E a generalized interior normal comes from *De Giorgi's blow-up construction* of E around a point of $\partial^* E$ in which, for $\varepsilon > 0$ small enough, one knows that $E \cap B(x,\varepsilon)$ is asymptotically close to the half-ball $\Pi^+_{\nu_E}(x) \cap B(x,\varepsilon)$. This construction will be taken up in more detail in preparation for Proposition 4.10 concerning the determination of normal traces of divergence measure fields. Moreover, the fundamental result of De Giorgi is that

$$|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial^* E,$$

which generalizes (2.4) to sets of finite perimeter and leads to De Giorgi's generalized Gauss–Green theorem (1.3). For the proof of these claims, we refer to [2, Theorem 3.59].

Crucial to the calculus on sets of finite perimeter E in Ω is Federer's structure theorem which we now recall. For any measurable set $E \subset \Omega$ and for any $\alpha \in [0,1]$ define the subsets

$$E^\alpha := \{x \in \Omega : d(E,x) = \alpha\}, \tag{2.5}$$

where

$$d(E,x) := \lim_{r\to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} \tag{2.6}$$

is the *Lebesgue density* of x in E. One calls E^1 and E^0 the *measure theoretic interior* and *exterior* of E in Ω, respectively, while $\partial^m E := \Omega \setminus (E^0 \cup E^1)$ is called the *measure theoretic boundary of* E in Ω. If E has finite perimeter in Ω, Federer's structure theorem (see [2, Theorem 3.61]) states that

$$\partial^* E \subset E^{1/2} \subset \partial^m E \tag{2.7}$$

and that there exists a subset $\mathcal{N}$ with $\mathscr{H}^{n-1}(\mathcal{N}) = 0$ such that

$$\Omega = E^1 \cup \partial^* E \cup E^0 \cup \mathcal{N}. \tag{2.8}$$

In particular, since $\mathscr{H}^{n-1}(\partial^m E \setminus \partial^* E) = 0$, we can integrate indifferently over $\partial^* E$ or $\partial^m E$ with respect to the Hausdorff measure $\mathscr{H}^{n-1}$ and E has density 0, $\frac{1}{2}$ or 1 in Ω at $\mathscr{H}^{n-1}$-a.e. $x \in E$. These facts will play an important role in Lemma 2.13 below on smooth approximations of χ_E for sets of finite perimeter.

In part to prepare for the approximation results in Lemma 2.13, we recall a few additional facts about BV functions. It is a well-known result from BV theory (see for instance [2, Corollary 3.80]) that every function of bounded variation u admits a representative which is the pointwise limit $\mathscr{H}^{n-1}$-a.e. of any mollification of u. In particular, this representative coincides $\mathscr{H}^{n-1}$-a.e. with the *precise representative* u^* of u defined by

$$u^*(x) := \begin{cases} \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy & \text{if this limit exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

and hence, given $u \in \mathrm{BV}(\Omega)$, if one defines $u_\varepsilon := u * \rho_\varepsilon$ in $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$, for any radially symmetric mollifier ρ, one has

$$u_\varepsilon(x) \to u^*(x) \quad \mathscr{H}^{n-1}\text{-a.e. } x \in \Omega. \tag{2.10}$$

Next, we record the following elementary extension property as a remark.

Remark 2.12. If $u \in \mathrm{BV}(\Omega)$ has compact support, then the zero extension $\hat{u}$ to $\mathbb{R}^n \setminus \Omega$ belongs to $\mathrm{BV}(\mathbb{R}^n)$. Indeed, it is clear that $\hat{u}$ is in $L^1(\mathbb{R}^n)$. Fix $\xi \in C_c^\infty(\Omega)$ with $\|\xi\|_\infty \le 1$ and $\xi = 1$ in a neighborhood of the support of u. Then, for any $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\varphi\|_\infty \le 1$ one has

$$\int_{\mathbb{R}^n} \hat{u}\,\mathrm{div}\,\varphi\,dx = \int_\Omega u\,\mathrm{div}\,\varphi\,dx = \int_\Omega u\,\mathrm{div}(\xi\varphi + (1-\xi)\varphi)\,dx = \int_\Omega u\,\mathrm{div}(\xi\varphi)\,dx \le |Du|(\Omega), \tag{2.11}$$

since $\xi\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$ and $\|\xi\varphi\|_\infty \le 1$. Taking the supremum over such φ, one obtains

$$|D\hat{u}|(\mathbb{R}^n) \le |Du|(\Omega) < \infty.$$

In addition, if $\varphi \in C_c^\infty(\Omega;\mathbb{R}^n)$, one has

$$\int_{\mathbb{R}^n} \hat{u}\,\mathrm{div}\,\varphi\,dx = \int_\Omega u\,\mathrm{div}\,\varphi\,dx,$$

which implies that $D\hat{u} = Du$ in $\mathcal{M}(\Omega;\mathbb{R}^n)$, since they are both finite Radon measures and $C_c^\infty(\Omega;\mathbb{R}^n)$ is dense in $C_c(\Omega;\mathbb{R}^n)$. Hence, one obtains $|D\hat{u}|(\Omega) = |Du|(\Omega)$, which, combined with (2.11) yields $|D\hat{u}|(\mathbb{R}^n \setminus \Omega) = 0$.

We conclude this subsection with the needed properties of mollifying characteristic functions of sets of finite perimeter.

Lemma 2.13. *Let $E \subset\subset \Omega$ be a set of finite perimeter in Ω and let $\chi_{E;\delta} := \chi_E * \rho_\delta$, where $\rho \in C_c^\infty(B(0,1))$ is a radially symmetric mollifier. Then the following results hold:*

(1) *There is a set $\mathcal{N}$ with $\mathscr{H}^{n-1}(\mathcal{N}) = 0$ such that, for all $x \in \Omega \setminus \mathcal{N}$, $\chi_{E;\delta}(x) \to \chi_E^*(x)$, where*

$$\chi_E^*(x) = \begin{cases} 1 & \text{if } x \in E^1, \\ \frac{1}{2} & \text{if } x \in \partial^* E, \\ 0 & \text{if } x \in E^0. \end{cases} \tag{2.12}$$

(2) *There exists $\delta_0 > 0$ such that for any $\delta < \delta_0$ one has the uniform bound*

$$\|\nabla\chi_{E;\delta}\|_{L^1(\Omega;\mathbb{R}^n)} \le |D\chi_E|(\Omega). \tag{2.13}$$

(3) *One has the following weak-star limits in $\mathcal{M}(\Omega;\mathbb{R}^n)$:*
   (a) $\nabla\chi_{E;\delta} \overset{*}{\rightharpoonup} D\chi_E$,
   (b) $\chi_E \nabla\chi_{E;\delta} \overset{*}{\rightharpoonup} \frac{1}{2} D\chi_E$,
   (c) $\chi_{\Omega\setminus E}\nabla\chi_{E;\delta} \overset{*}{\rightharpoonup} \frac{1}{2} D\chi_E$.

(4) *If $U \subset\subset \Omega$ is a an open set with $|D\chi_E|(\partial U) = 0$, then $|\nabla\chi_{E;\delta}|(U) \to |D\chi_E|(U)$.*

Proof. For the pointwise convergence of point (1), by (2.10), one knows that $\chi_{E;\delta} \to \chi_E^*$ $\mathscr{H}^{n-1}$-a.e. and that

$$\chi_E^*(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \chi_E(y)dy = d(E,x),$$

where $d(E,x)$ is the Lebesgue density (2.6). It follows that $\chi_E^*(x) = 1, 0$ if $x \in E^1, E^0$, respectively. Moreover, by (2.7), it follows that $\chi_E^*(x) = \frac{1}{2}$ if $x \in \partial^* E$.

For the estimate of point (2), consider first the case $\Omega = \mathbb{R}^n$. For any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_\infty \le 1$ one has

$$\int_{\mathbb{R}^n} \chi_{E;\delta}(x) \operatorname{div} \varphi(x)\, dx = \int_{\mathbb{R}^n} \chi_E(y) \operatorname{div}(\varphi * \rho_\delta)(y)\, dx \le |D\chi_E|(\mathbb{R}^n).$$

Taking the supremum over φ gives (2.13) in this case. In the general case, since $E \subset\subset \Omega$, there exists $\delta_0 > 0$ sufficiently small to ensure that for any $0 < \delta < \delta_0$ the support of $\chi_{E;\delta}$ is compact in Ω. Hence, if $\hat{\chi}_E$ denotes the zero extension to $\mathbb{R}^n$, one has $\|\nabla \hat{\chi}_{E;\delta}\|_{L^1(\Omega;\mathbb{R}^n)} = \|\nabla \hat{\chi}_{E;\delta}\|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \le |D\hat{\chi}_E|(\mathbb{R}^n)$ by the previous case. Remark 2.12 then shows that $|D\hat{\chi}_E|(\mathbb{R}^n) = |D\chi_E|(\Omega)$ and this gives (2.13) in the general case.

For the weak-star limit (a) of point (3), since $\chi_{E;\delta} \to \chi_E$ in $L^1(\Omega)$, one has

$$\int_{\mathbb{R}^n} \nabla \chi_{E;\delta} \cdot \varphi\, dx = -\int_{\mathbb{R}^n} \chi_{E;\delta} \operatorname{div} \varphi\, dx \to -\int_{\mathbb{R}^n} \chi_E \operatorname{div} \varphi\, dx = \int_{\mathbb{R}^n} \varphi \cdot dD\chi_E$$

for each $\varphi \in C_c^1(\Omega; \mathbb{R}^n)$. Consequently, one has the limit (a) in the sense of $\mathbb{R}^n$-vector-valued Radon measures, by the density of $C_c^1(\Omega; \mathbb{R}^n)$ in $C_c(\Omega; \mathbb{R}^n)$ with respect to the sup norm, and by the uniform boundedness of total variation given in (2.13).

In order to show limit (b), consider $\varphi \in C_c^1(\Omega; \mathbb{R}^n)$ and notice that

$$\begin{aligned}\int_\Omega \varphi\chi_E \cdot \nabla\chi_{E;\delta}\, dx &= \int_\Omega \chi_E \operatorname{div}(\chi_{E;\delta}\varphi)\, dx - \int_\Omega \chi_E\chi_{E;\delta} \operatorname{div}\varphi\, dx \\ &= -\int_\Omega \varphi\chi_{E;\delta} \cdot dD\chi_E - \int_\Omega \chi_E\chi_{E;\delta} \operatorname{div}\varphi\, dx.\end{aligned}$$

Now, let $\delta \to 0$ and apply Lebesgue's dominated convergence theorem to the measures $D\chi_E$ and $\mathscr{L}^n$ and use point (1) in order to obtain

$$\begin{aligned}\lim_{\delta\to 0} \int_\Omega \varphi\chi_E \cdot \nabla\chi_{E;\delta}\, dx &= -\int_\Omega \varphi\chi_E^* \cdot dD\chi_E - \int_\Omega \chi_E^2 \operatorname{div}\varphi\, dx \\ &= -\int_\Omega \frac{1}{2}\varphi \cdot dD\chi_E - \int_\Omega \chi_E \operatorname{div}\varphi\, dx \\ &= -\int_\Omega \frac{1}{2}\varphi \cdot dD\chi_E + \int_\Omega \varphi \cdot dD\chi_E\end{aligned}$$

since $\chi_E^* = \frac{1}{2}$ on $\partial^* E$ and $\operatorname{supp}|D\chi_E| = \partial^* E$. Therefore, by the density of $C_c^1(\Omega; \mathbb{R}^n)$ in $C_c(\Omega; \mathbb{R}^n)$ with respect to the supremum norm, claim (b) follows.

Finally, for limit (c), observe that

$$\chi_{\Omega\setminus E}\nabla\chi_{E;\delta} = \nabla\chi_{E;\delta} - \chi_E\nabla\chi_{E;\delta} \overset{*}{\rightharpoonup} \Big(1 - \frac{1}{2}\Big)D\chi_E$$

as $\delta \to 0$ by combining limits (a) and (b).

For property (4), we refer to [2, proof of Proposition 3.7]. □

2.4 Divergence measure fields and their fundamental properties

As a final preliminary, we give the precise definition of the class of low regularity vector fields that we will consider and present a few properties that are fundamental for the generalized Gauss–Green formulas and their applications. We begin with the class of vector fields.

Definition 2.14. Let $\Omega \subset \mathbb{R}^n$ be open and $1 \le p \le \infty$.

(a) A vector field $F \in L^p(\Omega; \mathbb{R}^n)$ is called a *divergence measure field*, and we write $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^n)$, if the distributional divergence $\operatorname{div} F$ is a real finite Radon measure on Ω.

(b) A vector field F is a *locally divergence measure field*, and we write $F \in \mathcal{DM}^p_{\text{loc}}(\Omega; \mathbb{R}^n)$, if $F_{|W} \in \mathcal{DM}^p(W; \mathbb{R}^n)$ for any $W \subset\subset \Omega$ open.

In the case $p = \infty$, F will be called a *(locally) essentially bounded divergence measure field.*

It is worth mentioning that if $F = (F_1, \dots, F_n)$ is a vector field with components $F_j \in \mathrm{BV}(\Omega) \cap L^p(\Omega)$, then $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^n)$; however, cancelations in the singular part of the measure $\operatorname{div} F$ can allow for $\mathcal{DM}^p(\Omega; \mathbb{R}^n)$ without having components in $\mathrm{BV}(\Omega) \cap L^p(\Omega)$.

A first important result concerns the absolute continuity properties of $\operatorname{div} F$ with respect to q-capacity, which depends on the Lebesgue index p for $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$. While this result is known (see [23, Theorem 2.8]), given its importance, a complete and self-contained proof using Definition 2.5 of the q-capacity will be given.

Theorem 2.15. *If $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ with $\frac{n}{n-1} < p \leq \infty$, then $|\operatorname{div} F| \ll \operatorname{cap}_q(\cdot, \Omega)$, where $q := \frac{p}{p-1}$ is the Hölder conjugate; that is, for each Borel set $B \subset \Omega$ such that $\operatorname{cap}_q(B, \Omega) = 0$, $|\operatorname{div} F|(B) = 0$.*

Proof. Since $\operatorname{div} F$ is a Radon measure on Ω, it follows that its positive and negative parts $(\operatorname{div} F)^{\pm}$ are well defined. Let $B \subset \Omega$ be a Borel set with $\operatorname{cap}_q(B, \Omega) = 0$. By the Hahn decomposition theorem, there exist Borel sets $B_{\pm} \subset B$ with $B_+ \cup B_- = B$ and $B_+ \cap B_- = \emptyset$ such that $\pm \operatorname{div} F \llcorner B_{\pm} \geq 0$; that is, $(\operatorname{div} F)^+ \llcorner B = \operatorname{div} F \llcorner B_+$ and $(\operatorname{div} F)^- \llcorner B = -\operatorname{div} F \llcorner B_-$. Hence, it suffices to prove that $\operatorname{div} F(B_{\pm}) = 0$, and, in order to do so, it suffices to prove $\operatorname{div} F(K) = 0$ for any compact subset K of $B_{\pm}$, by (2.1).

We show only the case $K \subset B_+$, as the case of B_- is analogous. By the monotonicity of capacity, (2.3), $\operatorname{cap}_q(K, \Omega) = 0$ for any $K \subset B$ if $\operatorname{cap}_q(B, \Omega) = 0$. Since $\operatorname{cap}_q(K, \Omega) = 0$ and $1 \leq q < n$, we can apply Lemma 2.8 in order to find a sequence of test functions $\varphi_j \in C^\infty_c(\Omega)$ such that
(1) $0 \leq \varphi_j \leq 1$ and $\varphi_j = 1$ on K,
(2) $\|\nabla \varphi_j\|_{L^q(\Omega; \mathbb{R}^n)} \to 0$,
(3) for each j, $\operatorname{supp}(\varphi_j)$ is contained in an open set $U_j \subset\subset \Omega$ such that $\{U_j\}$ is a decreasing sequence and $\bigcap_{j=1}^\infty U_j = K$.

Then, property (1) and the Hölder inequality yield

$$\operatorname{div} F(K) + \int_{\Omega \setminus K} \varphi_j \, d \operatorname{div} F = \int_{\Omega} \varphi_j \, d \operatorname{div} F = -\int_{\Omega} F \cdot \nabla \varphi_j \, dx \leq \|F\|_{L^p(U_1; \mathbb{R}^n)} \|\nabla \varphi_j\|_{L^q(\Omega; \mathbb{R}^n)}$$

and so, by properties (2) and (3),

$$\operatorname{div} F(K) \leq |\operatorname{div} F|(U_j \setminus K) + \|F\|_{L^p(U_1; \mathbb{R}^n)} \|\nabla \varphi_j\|_{L^q(\Omega; \mathbb{R}^n)} \to 0 \quad \text{as } j \to +\infty.$$ □

We remark that a similar result can be proven if $p = \frac{n}{n-1}$ by making use of the full $W^{1,q}$-capacity to extend Lemma 2.8 to this borderline case. We refer to [23, Theorem 2.8] for more details.

The following corollary (for which we refer also to [25, Theorem 3.2]), in the case $p = \infty$, is one of the pillars on which the proof of generalized Gauss–Green theorems for essential bounded divergence measure fields rests.

Corollary 2.16. *If $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $\frac{n}{n-1} \leq p < \infty$, then $|\operatorname{div} F|(B) = 0$ for any Borel set B with σ-finite $\mathscr{H}^{n-q}$ measure, where $q := \frac{p}{p-1}$. If $p = \infty$, then $|\operatorname{div} F| \ll \mathscr{H}^{n-1}$.*

Proof. If $\frac{n}{n-1} < p < \infty$, it suffices to apply Theorems 2.6 and 2.15. Indeed, one needs to show that $|\operatorname{div} F|(B) = 0$ for each $B \in \mathcal{B}(\Omega)$ such that there exists a family $\{B_j\} \subset \mathcal{B}(\Omega)$ satisfying $B \subset \bigcup_j B_j$ and $\mathscr{H}^{n-q}(B_j) < \infty$. For every compact $K \subset B_j$, one has $\mathscr{H}^{n-q}(K) < \infty$, hence $\operatorname{cap}_q(K, \Omega) = 0$ and thus $|\operatorname{div} F|(K) = 0$. By the inner regularity of the Radon measure $|\operatorname{div} F|$, we get $|\operatorname{div} F|(B_j) = 0$, and so $|\operatorname{div} F|(B) = 0$, by σ-subadditivity. For the case $p = \frac{n}{n-1}$, we refer to [25, Theorem 3.2]. The case $p = \infty$ follows analogously, by considering $B \in \mathcal{B}(\Omega)$ such that $\mathscr{H}^{n-1}(B) = 0$: then $\mathscr{H}^{n-1}(K) = 0$ for any compact $K \subset B$, which implies $\operatorname{cap}_1(K, \Omega) = 0$, and so $|\operatorname{div} F|(K) = 0$. The inner regularity yields $|\operatorname{div} F|(B) = 0$. □

The result of Corollary 2.16 is optimal. Indeed, we have the following result, due to Šilhavý [25, Example 3.3 and Proposition 6.1]. The underlying construction will also be discussed in Example 6.1 to illustrate the related fact of the possible absence of normal traces when $p < \infty$.

Proposition 2.17. *If $1 \leq p < \frac{n}{n-1}$, then for an arbitrary signed Radon measure with compact support μ there exists $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ such that $\operatorname{div} F = \mu$. This means that μ may be not absolutely continuous with respect to any Hausdorff measure or capacity.*

On the other hand, if $\frac{n}{n-1} \leq p \leq \infty$, then for any $s > n - q$ there exists a field $F \in \mathcal{DM}^p_{\rm loc}(\mathbb{R}^n; \mathbb{R}^n)$ such that $|\operatorname{div} F|$ is not $\mathscr{H}^s$ absolutely continuous.

It is not difficult to see that these results can be generalized to $\mu \in \mathcal{M}(\Omega)$ with compact support in Ω and $F \in \mathcal{DM}^p_{\rm loc}(\Omega; \mathbb{R}^n)$.

We now recall a product rule for essentially bounded divergence measure fields which is the second fundamental ingredient for the generalized Gauss–Green formulas. This result appeared in [5, Theorem 3.1] and we refer to [18, Theorem 2.1] for an improved proof.

Theorem 2.18. *Let $g \in BV(\Omega) \cap L^\infty(\Omega)$ and $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$. Then $gF \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$ and*

$$\operatorname{div}(gF) = g^* \operatorname{div} F + \overline{F \cdot Dg} \tag{2.14}$$

in the sense of Radon measures on Ω, where g^* is the precise representative of g (therefore, the limit of the mollified sequence $g_\delta := g * \rho_\delta$) and $\overline{F \cdot Dg}$ is a Radon measure, which is the weak-star limit of $F \cdot \nabla g_\delta$ and is absolutely continuous with respect to $|Dg|$.

In addition, if g is also locally Lipschitz, then

$$\operatorname{div}(gF) = g \operatorname{div} F + F \cdot \nabla g$$

in the sense of Radon measures on Ω.

As previously noted, in the proof of the Gauss–Green formulas this product rule will be applied directly along the lines of Vol'pert's treatment of essentially bounded BV fields. We now formalize a few relevant observations concerning the extension of Vol'pert's method to $\mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$-fields. See also the related Remarks 3.3 and 3.4.

Remark 2.19. As noted following Definition 2.14, one has $BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n) \subset \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$. This inclusion is strict for $n \geq 2$. Indeed, one might consider the classical example

$$F(x, y) = \sin\Big(\frac{1}{x - y}\Big)(1, 1) \in \mathcal{DM}^\infty(\mathbb{R}^2; \mathbb{R}^2) \setminus BV_{\rm loc}(\mathbb{R}^2; \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^2).$$

However, there is a certain parallelism between fields $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$ and functions $u \in BV(\Omega)$, as they enjoy many similar properties. For example, an important consequence of the coarea formula for BV functions (see [2, Theorem 3.40]) is the absolute continuity property $|Du| \ll \mathscr{H}^{n-1}$, while Corollary 2.16 yields $|\operatorname{div} F| \ll \mathscr{H}^{n-1}$. This property plays a fundamental role in the proof of Theorem 2.18, and, as we shall see in Section 3, it will be essential also in the proof of the Gauss–Green formulas. Moreover, $\mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$ is the natural extension of $BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ in the sense that (2.14) extends a similar product rule known for the latter space, which one can find in [29, Chapter 4, Section 6.4]. For the reader's convenience, we recall it here: *for any $F \in BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ and $g \in BV(\Omega) \cap L^\infty(\Omega)$, one has $gF \in BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ and, for any $j = 1, \dots, n$,*

$$D_j(gF_j) = g^* D_j F_j + F_j^* D_j g,$$

which implies

$$\operatorname{div}(gF) = g^* \operatorname{div} F + F^* \cdot Dg, \tag{2.15}$$

where F^* and g^* are the precise representatives of F and g.

It is quite easy to show that these product rules (2.14) and (2.15) are consistent if $F \in BV(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$. Indeed, one reasons as in point (3) of Lemma 2.13 concerning weak-star limits of gradients of mollified BV functions. Recall that $\overline{F \cdot Dg}$ is the weak-star limit of $F \cdot \nabla g_\delta$ as $\delta \to 0$, where g_δ is a mollification of g. Then one tests this sequence of Radon measures on a test function $\varphi \in C^1_c(\Omega)$ and some straightforward calculations yield

$$\int_\Omega \varphi \, d\overline{F \cdot Dg} = \int_\Omega \varphi F^* \cdot dDg.$$

The density of $C^1_c(\Omega)$ in $C_c(\Omega)$ implies the identity $\overline{F \cdot Dg} = F^* \cdot Dg$ in $\mathcal{M}(\Omega)$, and hence the consistency of the two product rules.

Finally, we note that Vol'pert's method consists of choosing $g = \chi_E$, where $E \subset\subset \Omega$ and applying the product rule to $\chi_E F$ and $\chi_E^2 F$ and then using a lemma on fields with compact support (see [29, Chapter 5, Section 1.4, Lemma 1]). We will follow the same path, using heavily Theorem 2.18 and Lemma 3.1 in the proof of Theorem 3.2.

We conclude this section with the following simple extension result for divergence measure fields, which is analogous to the zero extension result for BV functions given in Remark 2.12. Additional extension and gluing results will be given in Section 6.

Remark 2.20. If $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^n)$, for any $p \in [1, \infty]$, has compact support inside Ω, then its zero extension to all $\mathbb{R}^n$ is in $\mathcal{DM}^p(\mathbb{R}^n; \mathbb{R}^n)$. With

$$\hat{F}(x) := \begin{cases} F(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

one trivially has $\hat{F} \in L^p(\mathbb{R}^n; \mathbb{R}^n)$. Arguing as in Remark 2.12, one can show that $|\operatorname{div} \hat{F}|(\mathbb{R}^n) \leq |\operatorname{div} F|(\Omega) < \infty$ and so $\hat{F} \in \mathcal{DM}^p(\mathbb{R}^n; \mathbb{R}^n)$. In addition, if $\varphi \in C_c^\infty(\Omega)$, one obtains

$$\int_{\mathbb{R}^n} \hat{F} \cdot \nabla\varphi \, dx = \int_\Omega F \cdot \nabla\varphi \, dx,$$

which implies $\operatorname{div} \hat{F} = \operatorname{div} F$ in $\mathcal{M}(\Omega)$, since they are both finite Radon measures and $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$. Hence, one gets $|\operatorname{div} \hat{F}|(\Omega) = |\operatorname{div} F|(\Omega)$, which, combined with the above inequality, yields $|\operatorname{div} F|(\mathbb{R}^n \setminus \Omega) = 0$ and $\operatorname{div} \hat{F} = 0$ in $\mathbb{R}^n \setminus \Omega$.

Remark 2.21. One could obtain the same result by observing that, given a distribution $T \in \mathcal{D}'(\Omega)$, we have $\operatorname{supp}(\partial^\alpha T) \subset \operatorname{supp}(T)$ for any $\alpha \in \mathbb{N}^n$. Hence, for any $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ we have $\operatorname{supp}(\operatorname{div} F) \subset \operatorname{supp}(F)$. Thus, if F has compact support contained in an open set $V \subset\subset \Omega$, then $\operatorname{div} F = 0$ in $\Omega \setminus \overline{V}$.

3 Gauss–Green formulas and consistency of normal traces

In this section, we establish versions of the Gauss–Green formula for $\mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$ and $\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$-fields on sets of finite perimeter which are compactly contained in Ω. The method is analogous to the one Vol'pert used in order to prove his integration by parts theorem and it is based on the product rule established by Chen and Frid [5] and re-presented in [7] and [18]. The results are similar to those presented in the paper of Chen, Torres and Ziemer [9], but here we are not using their theory concerning the one-sided approximation of sets of finite perimeter by sets with smooth boundary. Therefore, we do not need to state a preliminary version of the theorem for open sets with smooth boundary. In addition, our approach can be easily generalized to any set of finite perimeter, even not compactly contained in Ω. Moreover, we will show the *consistency of normal traces* in the sense that if F is continuous on Ω then there is no jump component in the measure $\operatorname{div} F$ on $\partial^* E$ since the *interior and exterior normal traces* coincide and agree $\mathscr{H}^{n-1}$-a.e. with the classical dot product $F \cdot \nu_E$.

3.1 Gauss–Green formulas in $\mathcal{DM}^\infty$ and $\mathcal{DM}^\infty_{\mathrm{loc}}$

We begin with the following result concerning fields with compact support, which is valid for any $1 \leq p \leq \infty$ and can be seen as the easy case of the Gauss–Green formula, since there are no boundary terms.

Lemma 3.1. *Let $p \in [1, \infty]$. If $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^n)$ has compact support in Ω, then*

$$\operatorname{div} F(\Omega) = 0.$$

Proof. Since F has compact support, there exists an open set V satisfying $\operatorname{supp}(F) \subset V \subset\subset \Omega$. Then, by Remark 2.21, we have $\operatorname{div} F = 0$ in $\Omega \setminus \overline{V}$. Now, if we choose $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \equiv 1$ on a neighborhood of V, we obtain

$$0 = -\int_{\Omega\setminus V} F\cdot\nabla\varphi\,dx = -\int_{\Omega} F\cdot\nabla\varphi\,dx = \int_{\Omega}\varphi\,d\operatorname{div}F = \int_{\overline{V}}\varphi\,d\operatorname{div}F = \operatorname{div}F(\overline{V})$$

and hence $\operatorname{div}F(\Omega) = 0$. □

We next treat the case of essentially bounded divergence fields, where we recall that $\chi_{E;\delta} := \chi_E * \rho_\delta$, where $\rho \in C_c^\infty(B(0,1))$ is a radial mollifier.

Theorem 3.2. *Let $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ and let $E \subset\subset \Omega$ be a set of finite perimeter in Ω. Consider the signed Radon measures defined by*

$$\sigma_i := 2\overline{\chi_E F\cdot D\chi_E} \quad\text{and}\quad \sigma_e := 2\overline{\chi_{\Omega\setminus E}F\cdot D\chi_E}, \tag{3.1}$$

where $\overline{\chi_E F\cdot D\chi_E}$ and $\overline{\chi_{\Omega\setminus E}F\cdot D\chi_E}$ on Ω are the weak-star limits (up to subsequences) of $\chi_E F\cdot\nabla\chi_{E;\delta}$ and $\chi_{\Omega\setminus E}F\cdot\nabla\chi_{E;\delta}$ as $\delta\to 0$. The measures (3.1) are absolutely continuous with respect to $|D\chi_E|$ (and hence concentrated on ∂^*E) and satisfy

$$\operatorname{div}F(E^1) = -\sigma_i(\partial^*E) \quad\text{and}\quad \operatorname{div}F(E^1\cup\partial^*E) = -\sigma_e(\partial^*E). \tag{3.2}$$

The flux measures σ_i, σ_e admit Radon–Nikodym derivatives with respect to the measure $|D\chi_E| = \mathscr{H}^{n-1}\llcorner\partial^*E$ and denoting these derivatives by $(\mathcal{F}_i\cdot\nu_E), (\mathcal{F}_e\cdot\nu_E) \in L^1(\partial^*E;\mathscr{H}^{n-1})$ one has

$$\operatorname{div}F(E^1) = -\int_{\partial^*E}\mathcal{F}_i\cdot\nu_E\,d\mathscr{H}^{n-1} \quad\text{and}\quad \operatorname{div}F(E^1\cup\partial^*E) = -\int_{\partial^*E}\mathcal{F}_e\cdot\nu_E\,d\mathscr{H}^{n-1}. \tag{3.3}$$

Moreover, the normal traces $(\mathcal{F}_i\cdot\nu_E), (\mathcal{F}_e\cdot\nu_E)$ belong to $L^\infty(\partial^*E;\mathscr{H}^{n-1})$ and one has the estimates

$$\|\mathcal{F}_i\cdot\nu_E\|_{L^\infty(\partial^*E;\mathscr{H}^{n-1})} \le \|F\|_{L^\infty(E;\mathbb{R}^n)} \quad\text{and}\quad \|\mathcal{F}_e\cdot\nu_E\|_{L^\infty(\partial^*E;\mathscr{H}^{n-1})} \le \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}. \tag{3.4}$$

Proof. Using the product rule of Theorem 2.18, at the level of Radon measures on Ω, one has

$$\begin{aligned}\operatorname{div}(\chi_E^2F) &= \operatorname{div}(\chi_E(\chi_EF)) = \chi_E^*\operatorname{div}(\chi_EF) + \overline{\chi_EF\cdot D\chi_E}\\ &= \chi_E^*(\chi_E^*\operatorname{div}F + \overline{F\cdot D\chi_E}) + \overline{\chi_EF\cdot D\chi_E}\\ &= (\chi_E^*)^2\operatorname{div}F + \chi_E^*\overline{F\cdot D\chi_E} + \overline{\chi_EF\cdot D\chi_E},\end{aligned} \tag{3.5}$$

where χ_E^* is the precise representative of χ_E given in formula (2.12). On the other hand, one also has

$$\operatorname{div}(\chi_E^2F) = \operatorname{div}(\chi_EF) = \chi_E^*\operatorname{div}F + \overline{F\cdot D\chi_E} \tag{3.6}$$

and combining (3.5) with (3.6) yields

$$((\chi_E^*)^2 - \chi_E^*)\operatorname{div}F + \chi_E^*\overline{F\cdot D\chi_E} + \overline{\chi_EF\cdot D\chi_E} - \overline{F\cdot D\chi_E} = 0. \tag{3.7}$$

One has $|\operatorname{div}F| \ll \mathscr{H}^{n-1}$ by Corollary 2.16 and hence $\operatorname{div}F\llcorner(\partial^mE\setminus\partial^*E) = 0$. By formula (2.12), the first term in (3.7) satisfies

$$((\chi_E^*)^2 - \chi_E^*)\operatorname{div}F = -\frac{1}{4}\chi_{\partial^*E}\operatorname{div}F. \tag{3.8}$$

By Theorem 2.18, $|\overline{F\cdot D\chi_E}| \ll |D\chi_E|$ and $|\overline{\chi_EF\cdot D\chi_E}| \ll |D\chi_E|$ and therefore these two measures are also supported on ∂^*E. In particular, this implies that $\chi_E^*\overline{F\cdot D\chi_E} = \frac{1}{2}\overline{F\cdot D\chi_E}$. From this fact and (3.8) one obtains

$$\frac{1}{2}\chi_{\partial^*E}\operatorname{div}F + \overline{F\cdot D\chi_E} - 2\overline{\chi_EF\cdot D\chi_E} = 0. \tag{3.9}$$

Now, subtracting (3.9) from (3.6) gives

$$\begin{aligned}\operatorname{div}(\chi_EF) &= \chi_{E^1}\operatorname{div}F + \frac{1}{2}\chi_{\partial^*E}\operatorname{div}F + \overline{F\cdot D\chi_E} - \frac{1}{2}\chi_{\partial^*E}\operatorname{div}F + -\overline{F\cdot D\chi_E} + 2\overline{\chi_EF\cdot D\chi_E}\\ &= \chi_{E^1}\operatorname{div}F + 2\overline{\chi_EF\cdot D\chi_E}.\end{aligned}$$

On the other hand, adding (3.9) to (3.6) gives

$$\begin{aligned}\operatorname{div}(\chi_E F) &= \chi_{E^1}\operatorname{div} F + \frac{1}{2}\chi_{\partial^* E}\operatorname{div} F + \overline{F\cdot D\chi_E} + \frac{1}{2}\chi_{\partial^* E}\operatorname{div} F + \overline{F\cdot D\chi_E} - 2\overline{\chi_E F\cdot D\chi_E}\\ &= \chi_{E^1\cup\partial^* E}\operatorname{div} F + 2\overline{F\cdot D\chi_E} - 2\overline{\chi_E F\cdot D\chi_E}.\end{aligned}$$

Notice that $\overline{F\cdot D\chi_E} - \overline{\chi_E F\cdot D\chi_E}$ is the weak-star limit in $\mathcal{M}(\Omega)$ of a sequence

$$F\cdot\nabla(\chi_E * \rho_{\delta_k}) - \chi_E F\cdot\nabla(\chi_E * \rho_{\delta_k}) = (\chi_\Omega - \chi_E)F\cdot\nabla(\chi_E * \rho_{\delta_k}) = \chi_{\Omega\setminus E}F\cdot\nabla(\chi_E * \rho_{\delta_k})$$

and hence

$$\overline{F\cdot D\chi_E} - \overline{\chi_E F\cdot D\chi_E} = \overline{\chi_{\Omega\setminus E}F\cdot D\chi_E}. \tag{3.10}$$

One has then the following identities of Radon measures on Ω:

$$\operatorname{div}(\chi_E F) = \chi_{E^1}\operatorname{div} F + 2\overline{\chi_E F\cdot D\chi_E} \tag{3.11}$$

and

$$\operatorname{div}(\chi_E F) = \chi_{E^1\cup\partial^* E}\operatorname{div} F + 2\overline{\chi_{\Omega\setminus E}F\cdot D\chi_E}. \tag{3.12}$$

Since $\chi_E F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ clearly has compact support in Ω, by Lemma 3.1 and (3.11) one has

$$0 = \operatorname{div}(\chi_E F)(\Omega) = \operatorname{div} F(E^1) + 2\overline{\chi_E F\cdot D\chi_E}(\Omega).$$

Recalling that $\overline{\chi_E F\cdot D\chi_E}$ is supported on $\partial^* E$, one concludes that

$$\operatorname{div} F(E^1) = -2\overline{\chi_E F\cdot D\chi_E}(\Omega) = -2\overline{\chi_E F\cdot D\chi_E}(\partial^* E),$$

which is the interior Gauss–Green formula in (3.2) for σ_i defined in (3.1). In an analogous way, Lemma 3.1 and (3.12) yield

$$\operatorname{div} F(E^1\cup\partial^* E) = -2\overline{\chi_{\Omega\setminus E}F\cdot D\chi_E}(\partial^* E),$$

which is the exterior Gauss–Green formula in (3.2) for σ_e defined in (3.1).

Since $|\overline{\chi_E F\cdot D\chi_E}|$ and $|\overline{\chi_{\Omega\setminus E}F\cdot D\chi_E}|$ are absolutely continuous with respect to the measure

$$|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial^* E,$$

the Radon–Nikodym theorem implies that there exist functions $\mathcal{F}_i\cdot\nu_E$ and $\mathcal{F}_e\cdot\nu_E$ in $L^1(\partial^* E;\mathscr{H}^{n-1})$ such that

$$2\overline{\chi_E F\cdot D\chi_E} = (\mathcal{F}_i\cdot\nu_E)\mathscr{H}^{n-1}\llcorner\partial^* E \quad\text{and}\quad 2\overline{\chi_{\Omega\setminus E}F\cdot D\chi_E} = (\mathcal{F}_e\cdot\nu_E)\mathscr{H}^{n-1}\llcorner\partial^* E, \tag{3.13}$$

and hence one has the Gauss–Green formulas (3.3).

It remains only to justify estimates (3.4) on the L^∞-norm of the normal traces. The Lebesgue–Besicovitch differentiation theorem implies that, for $\mathscr{H}^{n-1}$-a.e. $x\in\partial^* E$, one has

$$(\mathcal{F}_i\cdot\nu_E)(x) = \lim_{r\to 0}\frac{2\overline{\chi_E F\cdot D\chi_E}(B(x,r))}{|D\chi_E|(B(x,r))}.$$

We claim that the family $|\chi_E F\cdot\nabla\chi_{E;\delta}|$ is uniformly bounded in $\mathcal{M}(\Omega)$ for $\delta > 0$ and small. Indeed,

$$\begin{aligned}|\chi_E F\cdot\nabla\chi_{E;\delta}|(\Omega) &:= \sup\Big\{\int_\Omega \varphi|\chi_E F\cdot\nabla\chi_{E;\delta}|\,dx : \varphi\in C_c^\infty(\Omega),\ \|\varphi\|_\infty\le 1\Big\}\\ &\le \int_\Omega |\chi_E F\cdot\nabla\chi_{E;\delta}|\,dx \le \|F\|_{L^\infty(E;\mathbb{R}^n)}\|\nabla\chi_{E;\delta}\|_{L^1(\Omega;\mathbb{R}^n)}\\ &\le \|F\|_{L^\infty(E;\mathbb{R}^n)}|D\chi_E|(\Omega),\end{aligned}$$

where the last inequality uses the bound (2.13).

Thus, there exists a weak-star converging subsequence, which we label with δ_k, and let the positive measure $\lambda_i \in \mathcal{M}(\Omega)$ be its limit. In an analogous way, we can prove that the family of Radon measures $|\chi_{\Omega \setminus E} F \cdot \nabla \chi_{E;\delta}|$ is uniformly bounded, we just need to put in the previous calculation the norm $\|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}$. So there exists a weak-star converging subsequence, which we label again with δ_k, whose limit is the positive Radon measure λ_e. Moreover, we observe that also the sequences $\chi_E|\nabla\chi_{E;\delta_k}|$ and $\chi_{\Omega\setminus E}|\nabla\chi_{E;\delta_k}|$ are bounded using the same argument as above. So there exist weak-star converging subsequences which we shall not relabel for simplicity of notation and which converge to positive measures $\mu_i, \mu_e \in \mathcal{M}(\Omega)$.

By Lemma 2.4, a sequence of balls $B(x, r_j)$ with $r_j \to 0$ can be chosen in such a way that

$$|D\chi_E|(\partial B(x, r_j)) = \lambda_i(\partial B(x, r_j)) = \mu_e(\partial B(x, r_j)) = 0.$$

Hence, by Lemmas 2.4 and 2.13 and because of $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial^* E$, we have

$$\begin{aligned}
\lim_{r_j\to 0}\left|\frac{2\overline{\chi_E F\cdot D\chi_E}(B(x,r_j))}{|D\chi_E|(B(x,r_j))}\right| &= \lim_{r_j\to 0}\left|\frac{\lim_{\delta_k\to 0} 2\int_{B(x,r_j)}\chi_E F\cdot\nabla\chi_{E;\delta_k}\,dx}{\lim_{\delta_k\to 0}\int_{B(x,r_j)}|\nabla\chi_{E;\delta_k}|\,dx}\right| \\
&\le \lim_{r_j\to 0}\frac{2\|F\|_{L^\infty(E;\mathbb{R}^n)}\lim_{\delta_k\to 0}\int_{B(x,r_j)}\chi_E|\nabla\chi_{E;\delta_k}|\,dx}{\lim_{\delta_k\to 0}\int_{B(x,r_j)}|\nabla\chi_{E;\delta_k}|\,dx} \\
&= 2\|F\|_{L^\infty(E;\mathbb{R}^n)}\lim_{r_j\to 0}\left(1-\frac{\lim_{\delta_k\to 0}\int_{B(x,r_j)}\chi_{\Omega\setminus E}|\nabla\chi_{E;\delta_k}|\,dx}{\lim_{\delta_k\to 0}\int_{B(x,r_j)}|\nabla\chi_{E;\delta_k}|\,dx}\right) \\
&\le 2\|F\|_{L^\infty(E;\mathbb{R}^n)}\lim_{r_j\to 0}\left(1-\frac{\lim_{\delta_k\to 0}|\int_{B(x,r_j)}\chi_{\Omega\setminus E}\nabla\chi_{E;\delta_k}\,dx|}{\lim_{\delta_k\to 0}\int_{B(x,r_j)}|\nabla\chi_{E;\delta_k}|\,dx}\right) \\
&= 2\|F\|_{L^\infty(E;\mathbb{R}^n)}\lim_{r_j\to 0}\left(1-\frac{1}{2}\frac{|D\chi_E(B(x,r_j))|}{|D\chi_E|(B(x,r_j))}\right) = \|F\|_{L^\infty(E;\mathbb{R}^n)}.
\end{aligned}$$

In the last equality we used the definition of reduced boundary: if $x \in \partial^* E$, then $|\nu_E|(x) = 1$, $|D\chi_E|(B(x, r)) > 0$ for $r > 0$ and $\nu_E(x) = \lim_{r\to 0}\frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$. This implies that

$$\lim_{r\to 0}\frac{|D\chi_E(B(x,r))|}{|D\chi_E|(B(x,r))} = |\nu_E(x)| = 1.$$

The estimate for the exterior normal trace $\mathcal{F}_e \cdot \nu_E$ can be obtained in a similar way, considering instead balls contained in Ω which satisfy $|D\chi_E|(\partial B(x, r_j)) = \lambda_e(\partial B(x, r_j)) = \mu_i(\partial B(x, r_j)) = 0$ and using the inequality

$$\left|\int_{B(x,r)}\chi_{\Omega\setminus E} F\cdot\nabla\chi_{E;\delta_k}\,dx\right| \le \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}\int_{B(x,r)}\chi_{\Omega\setminus E}|\nabla\chi_{E;\delta_k}|\,dx.$$

This completes the proof. □

Before proceeding with the first corollaries of Theorem 3.2, in the spirit of Remark 2.19, we would like to formalize a few remarks comparing the case of $\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ and $\mathrm{BV}(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n)$-fields.

Remark 3.3. Since the proof of Theorem 3.2 given above relies on the product rule for $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ and $g \in \mathrm{BV}(\Omega)\cap L^\infty(\Omega)$ and on Lemma 3.1, it follows from Remark 2.19 and [29, Lemma 1 in Chapter 5, Section 1.4] that Theorem 3.2 is consistent with Vol'pert's Gauss–Green formula for $\mathrm{BV}(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n)$-fields as given in [29, Chapter 5, Section 1.8]. In this particular case, one has $\mathcal{F}_i\cdot\nu_E = F_{\nu_E}\cdot\nu_E$ and $\mathcal{F}_e\cdot\nu_E = F_{-\nu_E}\cdot\nu_E$, where $F_{\pm\nu_E}(x)$ are the approximate limits of F in $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$ restricted to

$$\Pi^{\pm}_{\nu_E}(x) := \{y \in \mathbb{R}^n : (y - x)\cdot(\pm\nu_E(x)) \ge 0\};$$

that is, for any $\varepsilon > 0$ one has

$$\lim_{r\to 0}\frac{|\{y\in\mathbb{R}^n : |F(y) - F_{\pm\nu_E}(x)| \ge \varepsilon\}\cap B(x,r)\cap\Pi^{\pm}_{\nu_E}(x)|}{|B(x,r)|} = 0.$$

Remark 3.4. As a byproduct of identity (3.10) in the proof of Theorem 3.2, one has the following decomposition for the measure $\overline{F \cdot D\chi_E}$:

$$\overline{F \cdot D\chi_E} = \overline{\chi_E F \cdot D\chi_E} + \overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}. \tag{3.14}$$

It is, however, not possible in general to factorize the measures $\overline{\chi_E F \cdot D\chi_E}$ and $\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}$ into forms such as $\chi_E^* \overline{F \cdot D\chi_E}$ and $\chi_{\Omega\setminus E}^* \overline{F \cdot D\chi_E}$. For example, consider $E := [0,1]^n \subset\subset B(0,2) =: \Omega$ and $F(x) = H(x_1)e_1$, where $H(t) = \chi_{[0,+\infty)}(t)$ is the Heaviside function and $e_1 = (1, 0, \dots, 0)$ is the first element of the canonical basis of $\mathbb{R}^n$. Then it is not difficult to show that we have

$$\overline{\chi_E F \cdot D\chi_E} = \frac{1}{2} D_1\chi_E,$$
$$\overline{F \cdot D\chi_E} = D_1\chi_E - \frac{1}{2}\mathscr{H}^{n-1} \llcorner (\{0\} \times (0,1)^{n-1}),$$

which clearly implies $\overline{\chi_E F \cdot D\chi_E} \neq \frac{1}{2}\overline{F \cdot D\chi_E}$, but $\chi_E^* \overline{F \cdot D\chi_E} = \frac{1}{2}\overline{F \cdot D\chi_E}$, since $\chi_E^* = \frac{1}{2}$ on $\partial^* E$ by (2.12) and $\overline{F \cdot D\chi_E}$ is concentrated on $\partial^* E$, by Proposition 2.18. Thus,

$$\overline{\chi_E F \cdot D\chi_E} \neq \chi_E^* \overline{F \cdot D\chi_E}.$$

The inequality $\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} \neq (\chi_{\Omega\setminus E})^* \overline{F \cdot D\chi_E}$ follows easily from (3.14) and the previous inequality, since

$$\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} = \overline{F \cdot D\chi_E} - \overline{\chi_E F \cdot D\chi_E} \neq (1 - \chi_E^*)\overline{F \cdot D\chi_E} = (\chi_{\Omega\setminus E})^* \overline{F \cdot D\chi_E}.$$

An immediate corollary of Theorem 3.2 is a way to represent the measure $\operatorname{div} F$ on the reduced boundary of sets of finite perimeter compactly contained in the domain.

Corollary 3.5. *Let $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$. If $E \subset\subset \Omega$ is a set of finite perimeter in Ω, then*

$$\chi_{\partial^* E} \operatorname{div} F = 2\overline{\chi_E F \cdot D\chi_E} - 2\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E)\mathscr{H}^{n-1} \llcorner \partial^* E, \tag{3.15}$$

which implies

$$\operatorname{div} F(B) = \int_B (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E)\, d\mathscr{H}^{n-1} \tag{3.16}$$

for any Borel set $B \subset \partial^* E$, and

$$|\operatorname{div} F|(\partial^* E) = \int_{\partial^* E} |\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E|\, d\mathscr{H}^{n-1}. \tag{3.17}$$

Proof. Equation (3.15) follows immediately if one subtracts (3.11) from (3.12) and uses (3.13). Evaluating both measures in equation (3.15) over a Borel set B in $\partial^* E$ yields (3.16). Finally, (3.17) immediately follows from (3.15) and from properties of the total variation. □

The extension of the Gauss–Green formulas in Theorem 3.2 to locally essentially bounded divergence measure fields is straightforward. Indeed, if $E \subset\subset \Omega$, we can find an open set V satisfying $E \subset\subset V \subset\subset \Omega$. This simple topological fact allows us to state the following corollary for vector fields in $\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

Corollary 3.6. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and let $E \subset\subset \Omega$ be a set of finite perimeter in Ω. Then on a neighborhood of $\overline{E}$ one has internal and external flux measures defined by* (3.1) *and one has interior and exterior normal traces $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty(\partial^* E; \mathscr{H}^{n-1})$ such that formulas* (3.2), (3.3) *and* (3.15)–(3.17) *hold. In addition, one has the estimates*

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\partial^* E; \mathscr{H}^{n-1})} \leq \|F\|_{L^\infty(E;\mathbb{R}^n)} \quad \textit{and} \quad \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\partial^* E; \mathscr{H}^{n-1})} \leq \inf_V \{\|F\|_{L^\infty(V\setminus E;\mathbb{R}^n)}\}, \tag{3.18}$$

where the infimum is taken over all open sets V satisfying $E \subset\subset V \subset\subset \Omega$.

Proof. As noted above, there exists at least one open set V satisfying $E \subset\subset V \subset\subset \Omega$. Hence $F|_V \in \mathcal{DM}^\infty(V; \mathbb{R}^n)$ and $E \subset\subset V$, which means that one can apply Theorem 3.2 and Corollary 3.5. The two estimates in (3.18) follow similarly. □

3.2 Consistency of normal traces

As previously noted, for a general divergence measure field the measure $\operatorname{div} F$ contains a jump component at the boundary of a set of finite perimeter where the exterior and interior normal traces do not coincide. However, this does not happen if the field F is continuous. The following theorem is similar to [9, Theorem 7.2], however, our proof does not need the preliminary result given by [9, Lemma 7.1] and it is consequently more direct.

Theorem 3.7 (Consistency of the normal traces). *Let $F \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap C(\Omega; \mathbb{R}^n)$. If $E \subset\subset \Omega$ is a set of finite perimeter in Ω, then the interior and exterior normal traces coincide and admit a representative which is the classical dot product of F and the measure theoretic interior unit normal to E on $\partial^* E$. The Gauss–Green formulas* (3.3) *hence reduce to*

$$\operatorname{div} F(E^1) = -\int_{\partial^* E} F \cdot \nu_E \, d\mathscr{H}^{n-1} = \operatorname{div} F(E^1 \cup \partial^* E). \tag{3.19}$$

Proof. Up to taking an open set V such that $E \subset\subset V \subset\subset \Omega$, one can assume $F \in \mathcal{DM}^{\infty}(\Omega; \mathbb{R}^n)$. By Theorem 3.2, one has $2\overline{\chi_E F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E)\mathscr{H}^{n-1} \llcorner \partial^* E$ in the sense of Radon measures and $\mathcal{F}_i \cdot \nu_E \in L^{\infty}(\partial^* E; \mathscr{H}^{n-1})$. This means that for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$ one has

$$(\mathcal{F}_i \cdot \nu_E)(x) = \lim_{r \to 0} 2\frac{\overline{\chi_E F \cdot D\chi_E}(B(x, r))}{|D\chi_E|(B(x, r))}. \tag{3.20}$$

In addition, if $\chi_{E;\delta} := \chi_E * \rho_\delta$ is a mollification of χ_E, one knows that

$$\chi_E F \cdot \nabla \chi_{E;\delta} \overset{*}{\rightharpoonup} \overline{\chi_E F \cdot D\chi_E} \quad \text{in } \mathcal{M}(\Omega),$$

which means that, for all $\varphi \in C_c(\Omega)$,

$$\int_{\Omega} \varphi \chi_E F \cdot \nabla \chi_{E;\delta} \, dx \to \int_{\Omega} \varphi \, d\overline{\chi_E F \cdot D\chi_E} \quad \text{as } \delta \to 0.$$

Observe that $\varphi F \in C_c(\Omega; \mathbb{R}^n)$ and, since $\chi_E \nabla \chi_{E;\delta} \overset{*}{\rightharpoonup} \frac{1}{2} D\chi_E$, by point (3) (b) in Lemma 2.13, one also has

$$\int_{\Omega} (\varphi F) \cdot \nabla \chi_{E;\delta} \chi_E \, dx \to \int_{\Omega} (\varphi F) \cdot \frac{1}{2} dD\chi_E \quad \text{as } \delta \to 0.$$

Thus one can conclude that $\overline{\chi_E F \cdot D\chi_E} = \frac{1}{2} F \cdot D\chi_E$ in $\mathcal{M}(\Omega)$, which means that

$$2\overline{\chi_E F \cdot D\chi_E}(B(x, r)) = \int_{B(x,r)} F \cdot dD\chi_E = \int_{B(x,r)} F \cdot \nu_E \, d|D\chi_E|.$$

Moreover, by the continuity of F, the function $F \cdot \nu_E$ is well defined on $\partial^* E$ and is also in $L^1(\partial^* E; \mathscr{H}^{n-1})$. Thus, from (3.20), for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$, one obtains

$$(\mathcal{F}_i \cdot \nu_E)(x) = \lim_{r \to 0} \frac{\int_{B(x,r)} F(y) \cdot \nu_E(y) \, d|D\chi_E|(y)}{|D\chi_E|(B(x, r))} = F(x) \cdot \nu_E(x),$$

by the Lebesgue–Besicovitch differentiation theorem.

Applying the same steps to the measure $2\overline{\chi_{\Omega \setminus E} F \cdot D\chi_E}$ yields that it is equal to $F \cdot D\chi_E$ and hence one also finds that $\mathcal{F}_e \cdot \nu_E$ admits $F \cdot \nu_E$ as representative and hence it coincides with $\mathcal{F}_i \cdot \nu_E$ in the class of L^{∞}-functions. Finally, (3.19) follows easily from (3.3). □

From this theorem, we see that continuous divergence measure fields have no jump component in their distributional divergence.

Corollary 3.8. *Let $F \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap C(\Omega; \mathbb{R}^n)$. Then, for any $E \subset\subset \Omega$ set of finite perimeter in Ω, we have*

$$\chi_{\partial^* E} \operatorname{div} F = 0$$

in the sense of Radon measures.

Proof. It follows immediately from equation (3.15), from Corollary 3.6 and from Theorem 3.7. □

We remark that while this result says that $\chi_{\partial^* E}|\operatorname{div} F| = 0$ in the sense of Radon measures for any set $E \subset\subset \Omega$ of finite perimeter in Ω, we cannot strengthen this to obtain a better absolute continuity property of $\operatorname{div} F$ such as $|\operatorname{div} F| \ll \mathscr{H}^{n-t}$ for some $t \in [0, 1)$.

We also note that the L^∞-estimates in Theorem 3.2 (and so also those in Corollary 3.6) are sharp in the sense that we can find continuous divergence measure fields F for which

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\partial^* E;\mathscr{H}^{n-1})} = \|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\partial^* E;\mathscr{H}^{n-1})} = \|F\|_{L^\infty(E;\mathbb{R}^n)} = \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}$$

as the following simple example shows.

Example 3.9. Let $E = [0, 1]^n \subset\subset \Omega$ and let $F(x) = e_1 = (1, 0, \dots, 0)$. One has $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n) \cap C(\Omega; \mathbb{R}^n)$ and $\|F\|_{L^\infty(E;\mathbb{R}^n)} = \|F\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)} = 1$. Moreover, on $\{0\} \times (0, 1)^{n-1}$, $\nu_E = e_1$ and so over this part of $\partial^* E$ we have $\mathcal{F}_i \cdot \nu_E = \mathcal{F}_e \cdot \nu_E = F \cdot \nu_E = 1$, which implies the identity of the norms.

We conclude this section with a pair of remarks concerning normal traces.

Remark 3.10. We observe that in general the normal traces of an essentially bounded (but discontinuous) divergence measure field on the reduced boundary of a set of finite perimeter do not coincide $\mathscr{H}^{n-1}$-a.e. with the classical dot product. However, it has been shown that, roughly speaking, the normal traces coincide with the classical one on almost every surface. More precisely, let $I \subset \mathbb{R}$ be an open interval and let $\{\Sigma_t\}_{t\in I}$ be a family of oriented hypersurfaces in Ω such that there exist $\Omega' \subset\subset \Omega$, $\Phi \in C^1(\overline{\Omega'})$ and a family of open set $\Omega_t \subset\subset \Omega'$, $t \in I$, with $\Phi(\Omega') = I$, $\{\Phi = t\} = \Sigma_t = \partial\Omega_t$ for any $t \in I$, $|\nabla\Phi| > 0$ in Ω' and Σ_t is oriented by $\frac{\nabla\Phi}{|\nabla\Phi|}$. Then, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, we have

$$\mathcal{F}_i \cdot \nu_{\Omega_t} = \mathcal{F}_e \cdot \nu_{\Omega_t} = F \cdot \nu_{\Omega_t} \quad \mathscr{H}^{n-1}\text{-a.e. on } \Sigma_t, \text{ for } \mathscr{L}^1\text{-a.e. } t \in I.$$

For a proof of this result, see [1, Proposition 3.6] (although in that paper the definition of exterior normal trace is slightly different from ours, they are indeed equivalent by Proposition 4.10 below). We notice that in particular this statement applies to any family of balls $\{B(x_0, r)\}_{r\in(0,R)}$ inside Ω: indeed, in this case $I = (0, R)$ and $\Phi(x) = |x - x_0|^2$. Thus, for $\mathscr{L}^1$-a.e. $r \in (0, R)$, we have $|\operatorname{div} F|(\partial B(x_0, r)) = 0$,

$$\mathcal{F}_i \cdot \nu_{B(x_0,r)} = \mathcal{F}_e \cdot \nu_{B(x_0,r)} = -F \cdot \frac{(x - x_0)}{|x - x_0|} \quad \mathscr{H}^{n-1}\text{-a.e. on } \partial B(x_0, r)$$

and

$$\operatorname{div} F(B(x_0, r)) = \int_{\partial B(x_0,r)} F \cdot \frac{(x - x_0)}{|x - x_0|}\, d\mathscr{H}^{n-1}.$$

Remark 3.11. We notice that, by combining Remark 3.10, Theorem 3.2 and Corollary 3.6, one can recover the approximation result of Chen–Torres–Ziemer (as contained in [9, Theorem 5.2 (i) (b), (i) (g), (ii) (b), (ii) (g)]); that is, the integrals of the interior and the exterior normal traces over the reduced boundary are the limits of the integrals of the classical normal trace over the boundaries of a suitable family of smooth sets. Indeed, let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and let $E \subset\subset \Omega$ be a set of finite perimeter. Pick a smooth nonnegative radially symmetric mollifier $\rho \in C^\infty_c(B(0, 1))$ and consider the mollification $u_k(x) := (\chi_E * \rho_{\varepsilon_k})(x)$ of χ_E by some positive sequence $\varepsilon_k \to 0$. For $t \in (0, 1)$, one has $A_{k;t} := \{u_k > t\} \subset\subset \Omega$ if ε_k is small enough, following the notation of [9] and [10]. Since $|\operatorname{div} F| \ll \mathscr{H}^{n-1}$ (by Corollary 2.16), we can apply the approximation result stated in [9, Lemma 4.1] to the measure $\operatorname{div} F$ (see also [10, Theorem 3.1]) in order to obtain

$$\lim_{k\to+\infty} |\operatorname{div} F|(E^1 \Delta A_{k;t}) = 0 \quad \text{for } t \in \left(\frac{1}{2}, 1\right) \tag{3.21}$$

and

$$\lim_{k\to+\infty} |\operatorname{div} F|((E^1 \cup \partial^* E)\Delta A_{k;t}) = 0 \quad \text{for } t \in \left(0, \frac{1}{2}\right). \tag{3.22}$$

It is clear that the sets $A_{k;t}$ satisfy the hypothesis of Remark 3.10 for any k with $\Phi = u_k$, and so

$$\mathcal{F}_i \cdot \nu_{A_{k;t}} = \mathcal{F}_e \cdot \nu_{A_{k;t}} = F \cdot \nu_{A_{k;t}} \quad \mathscr{H}^{n-1}\text{-a.e. on } \partial A_{k;t}, \text{ for } \mathscr{L}^1\text{-a.e. } t \in (0, 1).$$

Now, since $A_{k;t}$ has a smooth boundary for $\mathscr{L}^1$-a.e. $t \in (0,1)$, it follows from Remark 2.10 that for these values of t one has $\mathscr{H}^{n-1}(\partial A_{k;t} \setminus \partial^* A_{k;t}) = 0$, and this implies $\mathscr{H}^{n-1}((A_{k;t})^1 \setminus A_{k;t}) = 0$. Hence, by Corollary 2.16 and the Gauss–Green formulas (3.3), one has

$$\operatorname{div} F(A_{k;t}) = -\int_{\partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} \tag{3.23}$$

for any $t \in (0,1) \setminus Z_k$, with $\mathscr{L}^1(Z_k) = 0$. Clearly, $Z := \bigcup_k Z_k$ is $\mathscr{L}^1$-negligible, and so (3.23) holds for any k and for any $t \in (0,1) \setminus Z$. Finally, one applies (3.3) to the set E and uses (3.21) and (3.22) to obtain

$$\lim_{k\to+\infty} \int_{\partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} = -\lim_{k\to+\infty} \operatorname{div} F(A_{k;t}) = -\operatorname{div} F(E^1) = \int_{\partial^* E} (\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1}$$

for $\mathscr{L}^1$-a.e. $t \in (\frac{1}{2}, 1)$, and

$$\lim_{k\to+\infty} \int_{\partial A_{k;t}} F \cdot \nu_{A_{k;t}} \, d\mathscr{H}^{n-1} = -\lim_{k\to+\infty} \operatorname{div} F(A_{k;t}) = -\operatorname{div} F(E^1 \cup \partial^* E) = \int_{\partial^* E} (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1}$$

for $\mathscr{L}^1$-a.e. $t \in (0, \frac{1}{2})$, which are the desired approximation results.

4 Integration by parts formulas and determination of normal traces

In this section, we make use of the Gauss–Green formulas to obtain integration by parts formulas and a few applications. In particular, the use of compactly supported test functions will lead us to an investigation of the local properties of normal traces of $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ on $\partial^* E$ for subsets $E \subset \Omega$ of locally finite perimeter and their complements. Moreover, we will show that the normal traces of F on $\partial^* E$ depend on E only though $\partial^* E$ and its orientation.

4.1 Integration by parts formulas

We begin with integration by parts formulas for a $\mathcal{DM}^\infty_{\mathrm{loc}}$-vector field and a Lipschitz scalar function over sets of finite perimeter compactly contained in the domain.

Theorem 4.1. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and let $E \subset\subset \Omega$ be a set of finite perimeter in Ω. Then, for any $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, we have*

$$\int_{E^1} \varphi \, d \operatorname{div} F = -\int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_E F \cdot \nabla \varphi \, dx \tag{4.1}$$

and

$$\int_{E^1 \cup \partial^* E} \varphi \, d \operatorname{div} F = -\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_E F \cdot \nabla \varphi \, dx. \tag{4.2}$$

Proof. As in the proof of Corollary 3.6, we take an open set U satisfying $E \subset\subset U \subset\subset \Omega$. Then $F|_U \in \mathcal{DM}^\infty(U; \mathbb{R}^n)$ and $\varphi \in \mathrm{Lip}(\overline{U})$, which implies also $\varphi \in W^{1,\infty}(U) \subset \mathrm{BV}(U) \cap L^\infty(U)$, since U is bounded. With a slight abuse of notation, from now on, we will write F instead of $F|_U$. By Theorem 2.18, we know that $\varphi F \in \mathcal{DM}^\infty(U; \mathbb{R}^n)$. Using the first Gauss–Green formula in (3.2) of Theorem 3.2, we obtain

$$\operatorname{div}(\varphi F)(E^1) = -2 \int_{\partial^* E} d\overline{\varphi \chi_E F \cdot D\chi_E}.$$

We have $\overline{\varphi \chi_E F \cdot D\chi_E} = \varphi \overline{\chi_E F \cdot D\chi_E}$: indeed, for any $\psi \in C_c(U)$,

$$\begin{aligned} \int_U \psi \, d\overline{\varphi \chi_E F \cdot D\chi_E} &= \lim_{\delta\to 0} \int_U \psi \, \varphi \chi_E F \cdot \nabla \chi_{E;\delta} \, dx \\ &= \lim_{\delta\to 0} \int_U (\psi\varphi) \chi_E F \cdot \nabla \chi_{E;\delta} \, dx = \int_U (\psi\varphi) \, d\overline{\chi_E F \cdot D\chi_E}, \end{aligned}$$

because $\psi\varphi \in C_c(U)$. Since $2\overline{\chi_E F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E)\mathscr{H}^{n-1} \llcorner \partial^* E$, it follows that

$$\int_{E^1} d\,\mathrm{div}(\varphi F) = -\int_{\partial^* E} \varphi(\mathcal{F}_i \cdot \nu_E)\, d\mathscr{H}^{n-1}. \tag{4.3}$$

On the other hand, Theorem 2.18 yields $\mathrm{div}(\varphi F) = \varphi\,\mathrm{div}\,F + F \cdot \nabla\varphi$, which implies

$$\int_{E^1} \varphi\, d\,\mathrm{div}\,F = -\int_{E^1} F \cdot \nabla\varphi\, dx + \int_{E^1} d\,\mathrm{div}(\varphi F). \tag{4.4}$$

Combining (4.3) with (4.4) and using $|E \Delta E^1| = 0$ yields (4.1). The proof of (4.2) is analogous and makes use of the second Gauss–Green formula in (3.2) of Theorem 3.2. □

More generally, it is also possible to remove the assumption $E \subset\subset \Omega$ if we localize with a Lipschitz function φ which is compactly supported in Ω.

Theorem 4.2. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and let $E \subset \Omega$ be a set of locally finite perimeter in Ω. Then there are well-defined interior and exterior normal traces of F on $\partial^* E$ satisfying $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty_{\mathrm{loc}}(\partial^* E; \mathscr{H}^{n-1})$ such that formulas* (4.1) *and* (4.2) *hold for any $\varphi \in \mathrm{Lip}_c(\Omega)$. In addition, for any open set $U \subset\subset \Omega$ one has the estimates*

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\partial^* E \cap U; \mathscr{H}^{n-1})} \le \|F\|_{L^\infty(E \cap U; \mathbb{R}^n)} \tag{4.5}$$

and

$$\|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\partial^* E \cap U; \mathscr{H}^{n-1})} \le \|F\|_{L^\infty(U \setminus E; \mathbb{R}^n)}. \tag{4.6}$$

Moreover, for any open set $U \subset\subset \Omega$,

$$\chi_{(\partial^* E) \cap U}\,\mathrm{div}\,F = (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E)\mathscr{H}^{n-1} \llcorner (\partial^* E \cap U), \tag{4.7}$$

which implies

$$\mathrm{div}\,F(B) = \int_B (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E)\, d\mathscr{H}^{n-1} \tag{4.8}$$

and

$$|\mathrm{div}\,F|(B) = \int_B |\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E|\, d\mathscr{H}^{n-1} \tag{4.9}$$

for any Borel set $B \subset\subset \Omega$ with $B \subset \partial^* E$.

Proof. We begin with the existence of the normal traces and the validity of formula (4.7). It is clear that there exists an open set $W \subset\subset \Omega$ such that $\mathrm{supp}(\varphi) \cap E \subset\subset W$ and for which we have then $F_{|W} \in \mathcal{DM}^\infty(W; \mathbb{R}^n)$ and $(\chi_E)_{|W} = \chi_{E \cap W} \in BV(W)$. This means that we can apply the Leibniz rule (Theorem 2.18) to $\varphi\chi_E F$ and $\varphi\chi_E^2 F$ and argue as in the proof of Theorem 3.2. We obtain

$$\mathrm{div}(\varphi\chi_E F) = \varphi\chi_E^*\,\mathrm{div}\,F + \varphi\overline{F \cdot D\chi_E} + \chi_E F \cdot \nabla\varphi, \tag{4.10}$$

$$\mathrm{div}(\varphi\chi_E^2 F) = \varphi(\chi_E^*)^2\,\mathrm{div}\,F + \varphi\chi_E^*\overline{F \cdot D\chi_E} + \varphi\overline{\chi_E F \cdot D\chi_E} + \chi_E F \cdot \nabla\varphi, \tag{4.11}$$

and we observe that $\mathrm{div}(\varphi\chi_E F) = \mathrm{div}(\varphi\chi_E^2 F)$, since $\chi_E^2 = \chi_E$. Making use of formulas (2.12) for χ_E^* and of decomposition (3.14) for the measure $\overline{F \cdot D\chi_E}$, formulas (4.10) and (4.11) become

$$\mathrm{div}(\varphi\chi_E F) = \varphi\chi_{E^1}\,\mathrm{div}\,F + \frac{1}{2}\varphi\chi_{\partial^* E}\,\mathrm{div}\,F + \varphi\overline{\chi_E F \cdot D\chi_E} + \varphi\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} + \chi_E F \cdot \nabla\varphi, \tag{4.12}$$

$$\mathrm{div}(\varphi\chi_E F) = \varphi\chi_{E^1}\,\mathrm{div}\,F + \frac{1}{4}\varphi\chi_{\partial^* E}\,\mathrm{div}\,F + \frac{3}{2}\varphi\overline{\chi_E F \cdot D\chi_E} + \frac{1}{2}\varphi\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} + \chi_E F \cdot \nabla\varphi. \tag{4.13}$$

Subtracting (4.13) from (4.12) gives the following identity between measures in $\mathcal{M}(W)$:

$$\varphi\chi_{\partial^* E}\,\mathrm{div}\,F = 2\varphi(\overline{\chi_E F \cdot D\chi_E} - \overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}). \tag{4.14}$$

Since $|\overline{\chi_E F \cdot D\chi_E}|, |\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}| \ll |D\chi_E|$ as measures in $\mathcal{M}(W)$ which are concentrated on $\partial^* E \cap W$, by the Radon–Nikodym theorem there exist two functions $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^1(\partial^* E \cap W; \mathscr{H}^{n-1})$ such that

$$2\overline{\chi_E F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E)\mathscr{H}^{n-1} \llcorner (\partial^* E \cap W) \quad \text{and} \quad 2\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} = (\mathcal{F}_e \cdot \nu_E)\mathscr{H}^{n-1} \llcorner (\partial^* E \cap W). \tag{4.15}$$

Now, with $U \subset\subset \Omega$ fixed, select $\varphi \in \mathrm{Lip}_c(\Omega)$ such that $U = \{\varphi \neq 0\}$, so that one has $\overline{U} \cap \partial^* E \subset \mathrm{supp}(\varphi) \cap \overline{E} \subset W$. Hence, for any $\psi \in C_c(U)$, one can take as test function $\Phi = \frac{\psi}{\varphi} \in C_c(U)$ in (4.14) to find

$$\int_U \psi \chi_{\partial^* E} \, d \operatorname{div} F = 2 \int_U \psi \, d(\overline{\chi_E F \cdot D\chi_E} - \overline{\chi_{\Omega \setminus E} F \cdot D\chi_E}).$$

This implies identity (4.7) in the sense of Radon measures. Formulas (4.8) and (4.9) are immediate consequences.

Next, we will show that (4.1) and (4.2) hold for any $\varphi \in \mathrm{Lip}_c(\Omega)$. Substituting into (4.12) the expression for $\varphi \chi_{\partial^* E} \operatorname{div} F$ given in (4.14), we find

$$\operatorname{div}(\varphi \chi_E F) = \varphi \chi_{E^1} \operatorname{div} F + \chi_E F \cdot \nabla \varphi + 2\varphi \overline{\chi_E F \cdot D\chi_E}, \tag{4.16}$$

to which we apply Lemma 3.1, using the fact that φ has compact support, in order to obtain (4.1). Analogously, substituting into (4.16) the expression for $2\varphi \overline{\chi_E F \cdot D\chi_E}$ which comes from (4.14), we find

$$\operatorname{div}(\varphi \chi_E^2 F) = \operatorname{div}(\varphi \chi_E F) = \varphi \chi_{E^1 \cup \partial^* E} \operatorname{div} F + \chi_E F \cdot \nabla \varphi + 2\varphi \overline{\chi_{\Omega \setminus E} F \cdot D\chi_E},$$

from which we deduce (4.2) in a similar way.

As for the L^∞-estimates, let $U \subset\subset \Omega$. Then we have $F_{|U} \in \mathcal{DM}^\infty(U; \mathbb{R}^n)$ and $\chi_{E \cap U} \in \mathrm{BV}(U)$. Hence, we obtain (3.4) for F and the set $E \cap U$, whose reduced boundary in U is $\partial^* E \cap U$, where $\partial^* E$ is the reduced boundary of E in Ω. Indeed, we notice that, in the last part of the proof of Theorem 3.2, the assumption $E \subset\subset \Omega$ is not necessary. Therefore, (4.5) and (4.6) follow. □

Before proceeding with some generalizations and applications of the integration by parts formulas, we wish to make some remarks about the normal traces in the extended context of $E \subset \Omega$ having only locally finite perimeter, as in Theorem 4.2.

Remark 4.3. It is possible to improve estimates (4.5) and (4.6) on the L^∞-norm of the normal traces. Indeed, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $E \subset \Omega$ is a set of locally finite perimeter in Ω, we can choose $U = (\partial E)_\varepsilon \cap V$, where $(\partial E)_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial E) < \varepsilon\}$ and $V \subset\subset \Omega$ is open. Then we get

$$\|\mathcal{F}_i \cdot \nu_E\|_{L^\infty(\partial^* E \cap V; \mathcal{H}^{n-1})} \leq \inf_{\varepsilon > 0} \{\|F\|_{L^\infty(E_\varepsilon; \mathbb{R}^n)}\},$$

where $E_\varepsilon := U \cap E = \{x \in E \cap V : \mathrm{dist}(x, \partial E) < \varepsilon\}$. On the other hand, a similar argument and (4.6) yield

$$\|\mathcal{F}_e \cdot \nu_E\|_{L^\infty(\partial^* E \cap V; \mathcal{H}^{n-1})} \leq \inf_{\varepsilon > 0} \{\|F\|_{L^\infty(E^\varepsilon; \mathbb{R}^n)}\},$$

where $E^\varepsilon := U \cap (\Omega \setminus E) = \{x \in (\Omega \setminus E) \cap V : \mathrm{dist}(x, \partial E) < \varepsilon\}$.

Remark 4.4. It is easy to see that, if $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \cap C(\Omega; \mathbb{R}^n)$ and E is a set of finite perimeter in Ω, then we have that the normal traces coincide with $F(x) \cdot \nu_E(x)$ for $\mathcal{H}^{n-1}$-a.e. $x \in (\partial^* E) \cap U$ and $\chi_{(\partial^* E) \cap U} \operatorname{div} F = 0$, for any open set $U \subset\subset \Omega$. Indeed, the traces are defined as the densities of the same Radon measures as in the case $E \subset\subset \Omega$.

As a first application of the integration by parts formulas, one can generalize the classical Green's identities to C^1-functions whose gradients are locally essentially bounded divergence measure fields.

Proposition 4.5. *Let $u \in C^1_c(\Omega)$ satisfy $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and let $E \subset \Omega$ be a set of finite perimeter in Ω. Then for each $v \in \mathrm{Lip}_c(\Omega)$ one has*

$$\int_{E^1} v \, d\Delta u = -\int_{\partial^* E} v \nabla u \cdot \nu_E \, d\mathcal{H}^{n-1} - \int_E \nabla v \cdot \nabla u \, dx, \tag{4.17}$$

and if $v \in C^1_c(\Omega)$ also satisfies $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, one has

$$\int_{E^1} v \, d\Delta u - u \, d\Delta v = -\int_{\partial^* E} (v \nabla u - u \nabla v) \cdot \nu_E \, d\mathcal{H}^{n-1}. \tag{4.18}$$

Moreover, if $E \subset\subset \Omega$, then one can drop the assumption that u and v have compact support in Ω.

Proof. We begin by noting that if $u \in C^1(\Omega)$ and $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, then $\nabla u \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \cap C(\Omega;\mathbb{R}^n)$. Thus, given a set E of finite perimeter in Ω, the normal traces of ∇u on $\partial^* E$ coincide with the classical dot product $\nabla u(x)\cdot \nu_E(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in (\partial^* E)\cap U$ and $\chi_{(\partial^* E)\cap U}\Delta u = 0$, for any open set $U \subset\subset \Omega$, by Remark 4.4.

Thus taking $u \in C^1_c(\Omega)$ such that $\Delta u \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ and taking $v \in \mathrm{Lip}_c(\Omega)$, for any set E of finite perimeter in Ω we have (4.17) by applying (4.1) of Theorem 4.2. If, in addition, $v \in C^1_c(\Omega)$ and satisfies $\Delta v \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, one also has (4.17) with the roles of u and v interchanged, which leads to (4.18). If $E \subset\subset \Omega$, one can appeal to Theorem 4.1 to eliminate the assumption on the compact support of u and v. □

We prove now a variant of the integration by parts formula in which the set of finite perimeter E and $\mathrm{supp}(\varphi)$ are not compactly contained in the domain Ω. This variant will be used in the applications of Section 5 on patching and extending divergence measure fields.

Proposition 4.6. *Let $V \subset\subset E^\circ \subset E \subset U$, where U, V are open sets and E is a set of finite perimeter in $\Omega := U \setminus V$, and let $F \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Then, for any $\varphi \in \mathrm{Lip}_c(U)$, we have*

$$\int_{E^0} \varphi \, d\,\mathrm{div}\, F = -\int_{\partial^* E} \varphi(\mathcal{F}_i \cdot \nu_{\Omega\setminus E})\, d\mathscr{H}^{n-1} - \int_{\Omega\setminus E} F\cdot\nabla\varphi\, dx, \tag{4.19}$$

$$\int_{E^0\cup\partial^* E} \varphi \, d\,\mathrm{div}\, F = -\int_{\partial^* E} \varphi(\mathcal{F}_e \cdot \nu_{\Omega\setminus E})\, d\mathscr{H}^{n-1} - \int_{\Omega\setminus E} F\cdot\nabla\varphi\, dx. \tag{4.20}$$

Proof. Let $\varphi \in \mathrm{Lip}_c(U)$. If we set $I_\varepsilon(V) = \{x \in U : \mathrm{dist}(x, V) < \varepsilon\}$, for some $\varepsilon > 0$ such that $\mathrm{dist}(I_\varepsilon(V), \partial E) > 0$, we can take a function $\eta \in C^\infty_c(I_\varepsilon(V))$ such that $\eta \equiv 1$ on $\overline{I_{\varepsilon/2}(V)}$. Now we define the function $\tilde\varphi := \varphi(1-\eta)$, so that we have $\tilde\varphi \in \mathrm{Lip}(\Omega)$, $\tilde\varphi = \varphi$ on $\overline{\Omega\setminus E}$ and $\tilde\varphi = 0$ on $\overline{I_{\varepsilon/2}(V)}$, hence $\tilde\varphi$ has compact support in Ω. Hence, we can apply Theorem 4.2 to F, $\Omega\setminus E$ and $\tilde\varphi$ in order to obtain

$$\int_{(\Omega\setminus E)^1} \tilde\varphi \, d\,\mathrm{div}\, F = -\int_{\partial^*(\Omega\setminus E)} \tilde\varphi(\mathcal{F}_i \cdot \nu_{\Omega\setminus E})\, d\mathscr{H}^{n-1} - \int_{\Omega\setminus E} F\cdot\nabla\tilde\varphi\, dx,$$

$$\int_{(\Omega\setminus E)^1\cup\partial^*(\Omega\setminus E)} \tilde\varphi \, d\,\mathrm{div}\, F = -\int_{\partial^*(\Omega\setminus E)} \tilde\varphi(\mathcal{F}_e \cdot \nu_{\Omega\setminus E})\, d\mathscr{H}^{n-1} - \int_{\Omega\setminus E} F\cdot\nabla\tilde\varphi\, dx.$$

By the properties of $\tilde\varphi$ and recalling that $(\Omega\setminus E)^1 = E^0$ and that $\partial^*(\Omega\setminus E) = \partial^* E$ (see (4.23) below), we deduce (4.19) and (4.20). □

Remark 4.7. When $\overline{E}$ is not compact in Ω, we cannot in general drop the assumption that φ has compact support in the integration by parts formulas, even if E is a set with globally finite perimeter measure on Ω. Indeed, if φ does not have compact support in Ω, we can take $\varphi = 1$. For example, consider $\Omega = \mathbb{R}^n \setminus \overline{B(0,\frac{1}{2})}$, $E = \mathbb{R}^n \setminus B(0,1)$ and $F = \frac{x}{|x|^n}$. It is clear that $|D\chi_E|$ is a finite Radon measure on Ω and that $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n) \cap C(\Omega;\mathbb{R}^n)$, with $\mathrm{div}\, F = 0$ and $(\mathcal{F}_i\cdot\nu_E)(x) = F(x)\cdot\frac{x}{|x|} = 1$ on $\partial B(0,1)$. If we take $\varphi = 1$ on Ω in (4.1), we have

$$0 = \mathrm{div}\, F(E^1) = -\int_{\partial B(0,1)} F(x)\cdot\frac{x}{|x|}\, d\mathscr{H}^{n-1}(x) = -\mathscr{H}^{n-1}(\partial B(0,1)) = -n\omega_n,$$

which is absurd.

4.2 Determination of normal traces

We begin by reinterpreting Theorem 4.2 in terms of the *normal trace functional* $(TF)_{\partial E} : \mathrm{Lip}_c(\Omega) \to \mathbb{R}$ defined by

$$(TF)_{\partial E}(\varphi) = \int_E F\cdot\nabla\varphi\, dx + \int_{E^1} \varphi\, d\,\mathrm{div}\, F. \tag{4.21}$$

This functional is well defined for any E of locally finite perimeter and for any $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ with $1 \le p \le \infty$ and has been well studied by Šilhavý in [25]. Theorem 4.2 says that when $p = \infty$ this functional can be represented by a locally essentially bounded function on $\partial^* E$ (the interior normal trace of F

on $\partial^* E$) in the sense that

$$(TF)_{\partial E}(\varphi) = -\int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1}, \tag{4.22}$$

from which it also follows that $\operatorname{supp}((TF)_{\partial E}) \subset \partial^* E$. On the other hand, if $p \neq \infty$, one cannot hope to find a representation like (4.22) with even $\mathcal{F}_i \cdot \nu_E \in L^1_{\mathrm{loc}}(\partial^* E; \mathscr{H}^{n-1})$, as Example 6.1 below illustrates.

In the case $p = \infty$, one might ask in what sense the normal traces depend on E. We will show that for sets of locally bounded perimeter, the normal traces are determined by $\partial^* E$ and its orientation, thus generalizing what is known for the case of E open, bounded with C^1-boundary (see [1, Proposition 3.2]). Our treatment begins by considering the normal traces on complementary sets.

If $E \subset \Omega$ has locally finite perimeter in Ω, then one knows that the complementary set $\Omega \setminus E$ also has locally finite perimeter in Ω, where

$$\partial^* (\Omega \setminus E) = \partial^* E \tag{4.23}$$

and

$$\nu_{\Omega \setminus E}(x) = -\nu_E(x) \quad \text{for all } x \in \partial^*(\Omega \setminus E) = \partial^* E.$$

Theorem 4.2 then shows that $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ also admits interior and exterior normal traces

$$(\mathcal{F}_i \cdot \nu_{\Omega \setminus E}), (\mathcal{F}_e \cdot \nu_{\Omega \setminus E}) \in L^\infty_{\mathrm{loc}}(\partial^* (\Omega \setminus E); \mathscr{H}^{n-1}),$$

with respect to $\partial^*(\Omega \setminus E)$, for which the integration by parts formulas (4.2) and (4.1) hold with $\Omega \setminus E$ in place of E. One easily obtains the following useful relations for normal traces on the boundary of complementary sets of locally finite perimeter in Ω.

Proposition 4.8. *If $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $E \subset \Omega$ is a set of locally finite perimeter in Ω, then*

$$(\mathcal{F}_e \cdot \nu_E) = -(\mathcal{F}_i \cdot \nu_{\Omega \setminus E}) \quad \mathscr{H}^{n-1}\text{-a.e. on } \partial^* E \tag{4.24}$$

and

$$(\mathcal{F}_e \cdot \nu_{\Omega \setminus E}) = -(\mathcal{F}_i \cdot \nu_E) \quad \mathscr{H}^{n-1}\text{-a.e. on } \partial^* E. \tag{4.25}$$

Proof. For any $\varphi \in C^1_c(\Omega)$, by Theorem 4.2 (using (4.2) on E and (4.1) on $\Omega \setminus E$), one has

$$\begin{aligned}
\int_\Omega F \cdot \nabla\varphi \, dx &= \int_E F \cdot \nabla\varphi \, dx + \int_{\Omega \setminus E} F \cdot \nabla\varphi \, dx \\
&= -\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_{E^1 \cup \partial^* E} \varphi \, d \operatorname{div} F - \int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu_{\Omega \setminus E}) \, d\mathscr{H}^{n-1} - \int_{E^0} \varphi \, d \operatorname{div} F \\
&= -\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu_{\Omega \setminus E}) \, d\mathscr{H}^{n-1} - \int_\Omega \varphi \, d \operatorname{div} F \\
&= -\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1} - \int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu_{\Omega \setminus E}) \, d\mathscr{H}^{n-1} + \int_\Omega F \cdot \nabla\varphi \, dx,
\end{aligned}$$

where one uses $(\Omega \setminus E)^1 = E^0$ and the facts that $\mathscr{H}^{n-1}(\Omega \setminus (E^0 \cup E^1 \cup \partial^* E)) = 0$ (by (2.8)) and $|\operatorname{div} F| \ll \mathscr{H}^{n-1}$ by Theorem 2.15. Hence for each $\varphi \in C^1_c(\Omega)$ one has

$$\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \nu_E) \, d\mathscr{H}^{n-1} = -\int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \nu_{\Omega \setminus E}) \, d\mathscr{H}^{n-1},$$

which gives (4.24) since $\varphi \in C^1_c(\Omega)$ is arbitrary, and by the density of $C^1_c(\Omega)$ in $C_c(\Omega)$. In a similar way, using (4.1) on E and (4.2) on $\Omega \setminus E$, one obtains (4.25). □

Remark 4.9. We notice that the L^∞-estimates are compatible with (4.24) and (4.25). Indeed, the L^∞-norm of F on $\Omega \setminus E$ controls both the L^∞-norm of the interior normal trace on $\Omega \setminus E$ and the L^∞-norm of the exterior normal trace on E. Analogously, $\|F\|_{L^\infty(E;\mathbb{R}^n)}$ controls $|\mathcal{F}_i \cdot \nu_E|$ and $|\mathcal{F}_e \cdot \nu_{\Omega \setminus E}|$.

We will now consider the normal traces of F on a common portion of the reduced boundary of two sets of locally finite perimeter. We will show that the traces agree if the measure theoretic normals are the same and have opposite signs if the measure theoretic normals have opposite orientation. Our proof will adapt that given in [1, Proposition 3.2] for bounded open sets with C^1-boundary.

For the proof, we need to recall a few additional facts from geometric measure theory. First, we recall a consequence of the basic comparison result between a positive Radon measure μ and k-dimensional Hausdorff measures through the use of k-dimensional densities of μ: if $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ with μ positive and $\mu \llcorner A = 0$ for a Borel set $A \subset \Omega$, then for each $k \geq 0$ one has

$$\mu(B(x,\rho)) = o(\rho^k) \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in A. \tag{4.26}$$

For a proof of this fact, see [2, Theorem 2.56]. Next, we recall elements of the structure of sets of locally finite perimeter given by De Giorgi's blow-up construction. If E is a set of locally finite perimeter in Ω, then for any $x \in \partial^* E$ one has

$$\chi_{(E-x)/\rho} \to \chi_{H^+_{\nu_E}(x)} \quad \text{and} \quad \chi_{((\Omega\setminus E)-x)/\rho} \to \chi_{H^-_{\nu_E}(x)} \quad \text{in } L^1(B(0,1)) \text{ as } \rho \to 0^+, \tag{4.27}$$

where $H^\pm_{\nu_E}(x) := \{y \in \mathbb{R}^n : \pm y \cdot \nu_E(x) \geq 0\}$. Moreover, the hyperplane $H_{\nu_E}(x) := \{y : y \cdot \nu_E(x) = 0\}$ is the approximate tangent space to the measure $\mathscr{H}^{n-1} \llcorner \partial^* E$ at $x \in \partial^* E$ in the sense that for any $\varphi \in C_c(\Omega)$ one has

$$\lim_{\rho \to 0^+} \rho^{-(n-1)} \int_{\partial^* E} \varphi\Big(\frac{y-x}{\rho}\Big)\, d\mathscr{H}^{n-1}(y) = \int_{H_{\nu_E}(x)} \varphi(z)\, d\mathscr{H}^{n-1}(z). \tag{4.28}$$

For the proof of these statements, see [2, Theorem 3.59].

Finally, let us consider two sets E_1, E_2 of locally finite perimeter in Ω. For $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E_1 \cap \partial^* E_2$, we have either $\nu_{E_1}(x) = \nu_{E_2}(x)$ or $\nu_{E_1}(x) = -\nu_{E_2}(x)$. This follows from the locality property of approximate tangent spaces, for which we refer to [2, Proposition 2.85 and Remark 2.87].

Proposition 4.10. *Let $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, and let E_1 and E_2 be sets of locally finite perimeter in Ω such that $\mathscr{H}^{n-1}(\partial^* E_1 \cap \partial^* E_2) \neq 0$. Then one has*

$$\mathcal{F}_i \cdot \nu_{E_1} = \mathcal{F}_i \cdot \nu_{E_2} \quad \textit{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = \mathcal{F}_e \cdot \nu_{E_2} \tag{4.29}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \partial^* E_1 \cap \partial^* E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$ and

$$\mathcal{F}_i \cdot \nu_{E_1} = -\mathcal{F}_e \cdot \nu_{E_2} \quad \textit{and} \quad \mathcal{F}_e \cdot \nu_{E_1} = -\mathcal{F}_i \cdot \nu_{E_2} \tag{4.30}$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \partial^* E_1 \cap \partial^* E_2 : \nu_{E_1}(y) = -\nu_{E_2}(y)\}$.

Proof. We begin with the first claim in (4.29). For $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E_1 \cap \partial^* E_2$ such that $\nu_{E_1}(x) = \nu_{E_2}(x)$ one has

$$x \text{ is a Lebesgue point for } \mathcal{F}_i \cdot \nu_{E_j} \text{ with respect to } \mathscr{H}^{n-1} \llcorner \partial^* E_j \text{ for } j = 1, 2 \tag{4.31}$$

and

$$|\operatorname{div} F|((E_1^1 \cup E_2^1) \cap B(x,\rho)) = o(\rho^{n-1}). \tag{4.32}$$

Indeed, the normal traces are in $L^\infty_{\mathrm{loc}}(\partial^* E; \mathscr{H}^{n-1})$ and so the Lebesgue–Besicovich differentiation theorem gives (4.31). For (4.32), it suffices to observe that $(E_1^1 \cup E_2^1) \cap \partial^* E_j = \emptyset$ for $j = 1, 2$, and so the property follows from (4.26) with $\mu = |\operatorname{div} F| \llcorner (E_1^1 \cup E_2^1)$ and $k = n - 1$.

Let $\eta \in C^\infty_c(B(0,1))$ and define $\eta_\rho(y) := \eta(\frac{y-x}{\rho})$. By the integration by parts formula (Theorem 4.2), one has

$$\int_{E_j^1} \eta_\rho \, d \operatorname{div} F = -\int_{\partial^* E_j} \eta_\rho (\mathcal{F}_i \cdot \nu_{E_j})\, d\mathscr{H}^{n-1} - \int_{E_j} F \cdot \nabla \eta_\rho \, dy \tag{4.33}$$

for $j = 1, 2$. Using (4.32), one sees that

$$\bigg| \int_{E_1^1} \eta_\rho \, d \operatorname{div} F - \int_{E_2^1} \eta_\rho \, d \operatorname{div} F \bigg| \leq |\operatorname{div} F|((E_1^1 \cup E_2^1) \cap B(x,\rho)) = o(\rho^{n-1}). \tag{4.34}$$

Since $\nabla \eta_\rho = \frac{1}{\rho}(\nabla \eta)_\rho$, one also has

$$\left| \int_{E_1} F \cdot \nabla \eta_\rho \, dy - \int_{E_2} F \cdot \nabla \eta_\rho \, dy \right| \leq \frac{1}{\rho} \|F\|_{L^\infty(B(x,1);\mathbb{R}^n)} \|\nabla \eta\|_{L^\infty(B(0,1);\mathbb{R}^n)} |(E_1 \Delta E_2) \cap B(x, \rho)|. \tag{4.35}$$

Next, observe that

$$\begin{aligned} \rho^{-n} |(E_1 \Delta E_2) \cap B(x, \rho)| &= \rho^{-n} \int_{B(x,\rho)} |\chi_{E_1} - \chi_{E_2}| \, dy \\ &= \int_{B(0,1)} |\chi_{E_1}(x + \rho z) - \chi_{E_2}(x + \rho z)| \, dz \\ &= \int_{B(0,1)} |\chi_{\frac{E_1 - x}{\rho}}(z) - \chi_{\frac{E_2 - x}{\rho}}(z)| \, dz \to 0 \end{aligned}$$

as $\rho \to 0$, where one uses (4.27) and the fact that $H^+_{\nu_{E_1}}(x) = H^+_{\nu_{E_2}}(x)$. Hence, (4.35) implies

$$\left| \int_{E_1} F \cdot \nabla \eta_\rho \, dy - \int_{E_2} F \cdot \nabla \eta_\rho \, dy \right| = o(\rho^{n-1}). \tag{4.36}$$

Subtracting (4.33) with $j = 2$ from (4.33) with $j = 1$ and using (4.34) and (4.36), one obtains

$$\int_{\partial^* E_1} \eta_\rho (\mathcal{F}_i \cdot \nu_{E_1}) \, d\mathscr{H}^{n-1} - \int_{\partial^* E_2} \eta_\rho (\mathcal{F}_i \cdot \nu_{E_2}) \, d\mathscr{H}^{n-1} = o(\rho^{n-1}). \tag{4.37}$$

On the other hand, since x is a Lebesgue point for $\mathcal{F}_i \cdot \nu_{E_j}$ with respect to $\mathscr{H}^{n-1} \llcorner \partial^* E_j$, one has

$$\begin{aligned} &\left| \int_{\partial^* E_j} \eta_\rho (\mathcal{F}_i \cdot \nu_{E_j}) \, d\mathscr{H}^{n-1} - (\mathcal{F}_i \cdot \nu_{E_j})(x) \int_{\partial^* E_j} \eta_\rho \, d\mathscr{H}^{n-1} \right| \\ &\qquad \leq \int_{\partial^* E_j} \eta_\rho(y) |(\mathcal{F}_i \cdot \nu_{E_j})(y) - (\mathcal{F}_i \cdot \nu_{E_j})(x)| \, d\mathscr{H}^{n-1}(y) = o(\rho^{n-1}) \end{aligned} \tag{4.38}$$

for $j = 1, 2$. In addition, (4.28) implies that

$$\left| \rho^{-(n-1)} \int_{\partial^* E_j} \eta_\rho \, d\mathscr{H}^{n-1} - \int_{H_{\nu_{E_j}}(x)} \eta \, d\mathscr{H}^{n-1} \right| = o(1), \tag{4.39}$$

for $j = 1, 2$. Hence, by (4.38), (4.39) and the triangle inequality, one has

$$\left| \rho^{-(n-1)} \int_{\partial^* E_j} \eta_\rho (\mathcal{F}_i \cdot \nu_{E_j}) \, d\mathscr{H}^{n-1} - (\mathcal{F}_i \cdot \nu_{E_j})(x) \int_{H_{\nu_{E_j}}(x)} \eta \, d\mathscr{H}^{n-1} \right| = o(1).$$

Hence, for $j = 1, 2$ one has

$$\rho^{-(n-1)} \int_{\partial^* E_j} \eta_\rho (\mathcal{F}_i \cdot \nu_{E_j}) \, d\mathscr{H}^{n-1} \to (\mathcal{F}_i \cdot \nu_{E_j})(x) \int_{H_{\nu_{E_j}}(x)} \eta \, d\mathscr{H}^{n-1} \quad \text{as } \rho \to 0. \tag{4.40}$$

Now choose η such that $\eta \geq \frac{1}{2}$ on $H_{\nu_{E_j}}(x) \cap B(0, \frac{1}{2})$ so that the integral over $H_{\nu_{E_j}}(x)$ is not zero. By recalling that $H_{\nu_{E_1}}(x) = H_{\nu_{E_2}}(x)$, formulas (4.37) and (4.40) imply $(\mathcal{F}_i \cdot \nu_{E_1})(x) = (\mathcal{F}_i \cdot \nu_{E_2})(x)$.

As for the other identities, notice that (4.24) gives $(\mathcal{F}_e \cdot \nu_{E_j}) = -(\mathcal{F}_i \cdot \nu_{\Omega \setminus E_j})$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E_j$, for $j = 1, 2$. Moreover, since $\nu_{\Omega \setminus E_j} = -\nu_{E_j}$ $\mathscr{H}^{n-1}$-a.e. on $\partial^* E_j$ and $\nu_{E_1}(x) = \nu_{E_2}(x)$, one has $\nu_{\Omega \setminus E_1}(x) = \nu_{\Omega \setminus E_2}(x)$. Since $\Omega \setminus E_j$ is a set of locally finite perimeter in Ω, one can apply the identity we just proved to obtain

$$(\mathcal{F}_e \cdot \nu_{E_1})(x) = -(\mathcal{F}_i \cdot \nu_{\Omega \setminus E_1})(x) = -(\mathcal{F}_i \cdot \nu_{\Omega \setminus E_2})(x) = (\mathcal{F}_e \cdot \nu_{E_2})(x)$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \{y \in \partial^* E_1 \cap \partial^* E_2 : \nu_{E_1}(y) = \nu_{E_2}(y)\}$, which is the second claim in (4.29). The identities of (4.30) follow in an analogous way by using formulas (4.24)–(4.25) and the previous argument applied to E_1 and $\Omega \setminus E_2$. □

5 Gluing constructions and extension theorems

In this section, we will present two gluing constructions for building $\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$-fields from a pair of $\mathcal{DM}^\infty$-fields whose domains decompose Ω. The first construction involves subdomains whose overlap is an open subset containing the boundary ∂E of a bounded set of finite perimeter in Ω and the gluing takes place along ∂E by restriction of the respective fields to E and its complement. The second construction involves complementary subsets, one of which is an open bounded subset U whose topological boundary has finite $\mathscr{H}^{n-1}$-measure. The pair of fields are extended by zero on their complements and summed to give the gluing along ∂U. Since there are no a priori compatibility assumptions made on the pair of fields, the results provide a wealth of $\mathcal{DM}^\infty$ extensions of a given $\mathcal{DM}^\infty$-field. The two theorems presented here are similar to [7, Theorem 3] and [9, Theorem 8.5 and Corollary 8.6], respectively; however, we have removed some of their assumptions on domains and modified and completed the proofs. In particular, we make use of the integration by parts formula on the complement of sets of finite perimeter compactly contained in the domain (Proposition 4.6) which justifies the treatment of the term with an unbounded domain in [9, Corollary 8.6]. In addition, we have refined the conclusions by providing representation formulas for the jump components of the distributional divergence of the fields constructed and given L^∞-estimates of the relevant normal traces. Finally, we use the second construction to obtain Gauss–Green and integration by parts formulas up to the boundary of a bounded domain U such that $\mathscr{H}^{n-1}(\partial U \setminus \partial^* U) = 0$.

We begin with the extension theorem with overlapping domains.

Theorem 5.1. *Let $W \subset\subset E^\circ \subset E \subset\subset U \subset \Omega$, where Ω, U and W are open sets and E is a set of finite perimeter in Ω. Let $F_1 \in \mathcal{DM}^\infty(U;\mathbb{R}^n)$ and $F_2 \in \mathcal{DM}^\infty(\Omega \setminus \overline{W};\mathbb{R}^n)$. Then*

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in E, \\ F_2(x) & \text{if } x \in \Omega \setminus E, \end{cases}$$

belongs to $\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$, and

$$\begin{aligned}\|F\|_{\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)} &= \max\{\|F_1\|_{L^\infty(E;\mathbb{R}^n)}, \|F_2\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}\} + |\operatorname{div} F|(E^1) \\ &\quad + |\operatorname{div} F_2|(E^0) + \|\mathcal{F}_{i,1}\cdot\nu_E - \mathcal{F}_{e,2}\cdot\nu_E\|_{L^1(\partial^*E;\mathscr{H}^{n-1})},\end{aligned}$$

where $\mathcal{F}_{i,1}\cdot\nu_E$ is the interior normal trace of F_1 over ∂^*E and $\mathcal{F}_{e,2}\cdot\nu_E$ is the exterior normal trace of F_2 over ∂^*E. In addition, we have

$$\operatorname{div} F = \chi_{E^1}\operatorname{div} F_1 + \chi_{E^0}\operatorname{div} F_2 + (\mathcal{F}_{i,1}\cdot\nu_E - \mathcal{F}_{e,2}\cdot\nu_E)\mathscr{H}^{n-1} \llcorner \partial^*E \tag{5.1}$$

in the sense of Radon measures on Ω, which in particular implies the following representation for the jump component:

$$\chi_{\partial^*E}\operatorname{div} F = (\mathcal{F}_{i,1}\cdot\nu_E - \mathcal{F}_{e,2}\cdot\nu_E)\mathscr{H}^{n-1} \llcorner \partial^*E. \tag{5.2}$$

We notice that we recover [7, Theorem 3] if we take $\Omega = \mathbb{R}^n$ and U bounded.

Proof. Obviously, $F \in L^\infty(\Omega;\mathbb{R}^n)$ and

$$\|F\|_{L^\infty(\Omega;\mathbb{R}^n)} = \max\{\|F_1\|_{L^\infty(E;\mathbb{R}^n)}, \|F_2\|_{L^\infty(\Omega\setminus E;\mathbb{R}^n)}\}.$$

By applying the integration by parts formulas (4.1) to E and (4.19) to $\Omega \setminus E$, for each $\varphi \in C_c^1(\Omega)$ with $\|\varphi\|_\infty \leq 1$ one has

$$\begin{aligned}\int_\Omega F\cdot\nabla\varphi\,dx &= \int_E F_1\cdot\nabla\varphi\,dx + \int_{\Omega\setminus E} F_2\cdot\nabla\varphi\,dx \\ &= -\int_{E^1}\varphi\,d\operatorname{div} F_1 - \int_{E^0}\varphi\,d\operatorname{div} F_2 - \int_{\partial^*E}(\mathcal{F}_{i,1}\cdot\nu_E + \mathcal{F}_{i,2}\cdot\nu_{\Omega\setminus E})\varphi\,d\mathscr{H}^{n-1} \qquad (5.3) \\ &\leq |\operatorname{div} F_1|(E^1) + |\operatorname{div} F_2|(E^0) + \|\mathcal{F}_{i,1}\cdot\nu_E + \mathcal{F}_{i,2}\cdot\nu_{\Omega\setminus E}\|_{L^1(\partial^*E;\mathscr{H}^{n-1})}.\end{aligned}$$

Thus, taking the supremum over φ on the left-hand side, one has $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$.

Now, by (5.3) and (4.24), for any $\varphi \in C_c^1(\Omega)$ one has

$$\int_\Omega \varphi \, d \operatorname{div} F = -\int_\Omega F \cdot \nabla\varphi \, dx = \int_{E^1} \varphi \, d \operatorname{div} F_1 + \int_{E^0} \varphi \, d \operatorname{div} F_2 + \int_{\partial^* E} (\mathcal{F}_{i,1} \cdot \nu_E - \mathcal{F}_{e,2} \cdot \nu_E)\varphi \, d\mathscr{H}^{n-1},$$

from which identities (5.1) and (5.2) follow. Finally, basic properties of the total variation then yield the estimate on $\|F\|_{\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)}$. □

Before turning our attention to the extension theorem for complementary domains, we will need a result from measure theory which allows us to approximate open sets with finite boundary measure from the inside and from the outside. A similar result is contained in [9, Proposition 8.1] in order to prove their extension Theorem 8.5; however, in [9] only the interior approximation is considered. In order to prove the result, we follow the line of the proof of [2, Proposition 3.62].

Proposition 5.2. *Let $U \subset \mathbb{R}^n$ be a bounded open set with $\mathscr{H}^{n-1}(\partial U) < \infty$. Then there exist two sequences of open bounded sets U_k and W_k such that $U_k \subset\subset U \subset\subset W_k$ and*
(1) $|U \setminus U_k| \to 0$,
(2) $\limsup_{k\to+\infty} \mathscr{H}^{n-1}(\partial U_k) \leq 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \mathscr{H}^{n-1}(\partial U)$,
(3) $|W_k \setminus U| \to 0$,
(4) $\limsup_{k\to+\infty} \mathscr{H}^{n-1}(\partial W_k) \leq 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \mathscr{H}^{n-1}(\partial U)$.

Proof. By the definition of Hausdorff measure, for each integer k, there exists a δ_k-covering of ∂U by closed sets $\{C_j\}$ satisfying $\operatorname{diam}(C_j) =: 2r_j < \delta_k$, $\partial U \subset \bigcup_{j=1}^\infty C_j$ and

$$\sum_{j=1}^\infty \omega_{n-1} r_j^{n-1} \leq \mathscr{H}_{\delta_k}^{n-1}(\partial U) + \frac{1}{k} \leq \mathscr{H}^{n-1}(\partial U) + \frac{1}{k}. \tag{5.4}$$

It is clear that we can cover ∂U with a family of balls $\{B(x_j, 2r_j)\}$, for some $x_j \in C_j$. Since ∂U is compact, there exists a finite covering $\{B(x_j, 2r_j)\}_{j=1}^{m_k}$ and so we set $V_k := \bigcup_{j=1}^{m_k} B(x_j, 2r_j)$. We observe that

$$\partial V_k \subset \bigcup_{j=1}^{m_k} \partial B(x_j, 2r_j).$$

This inclusion implies

$$\mathscr{H}^{n-1}(\partial V_k) \leq 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \sum_{j=1}^{m_k} \omega_{n-1} r_j^{n-1},$$

which, together with (5.4), yields

$$\mathscr{H}^{n-1}(\partial V_k) \leq 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \Big(\mathscr{H}^{n-1}(\partial U) + \frac{1}{k} \Big) \quad \text{for all } k. \tag{5.5}$$

We set $U_k := U \setminus \overline{V_k}$ and so, by (5.4), we have

$$|U \setminus U_k| = |U \cap V_k| \leq |V_k| \leq \sum_{j=1}^{m_k} \omega_n r_j^n < \frac{\delta_k}{2} \frac{\omega_n}{\omega_{n-1}} \sum_{j=1}^{m_k} \omega_{n-1} r_j^{n-1} \leq \frac{\delta_k}{2} \frac{\omega_n}{\omega_{n-1}} \Big(\mathscr{H}^{n-1}(\partial U) + \frac{1}{k} \Big),$$

which goes to zero as $\delta_k \to 0$. Finally, $\partial U_k = \partial V_k \cap U$ and so (5.5) implies point (2).

For the exterior approximation we choose $W_k := U \cup V_k$, which is clearly bounded. It is easy to see that we have $\partial W_k = \partial V_k \setminus \overline{U}$ and

$$|W_k \setminus U| = |V_k \setminus U| \leq |V_k| \to 0.$$

It follows also that $\mathscr{H}^{n-1}(\partial W_k) \leq \mathscr{H}^{n-1}(\partial V_k)$, which, by (5.5), implies

$$\limsup_{k\to+\infty} \mathscr{H}^{n-1}(\partial W_k) \leq 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \mathscr{H}^{n-1}(\partial U),$$

as desired. □

We now are ready for the extension theorem with respect to complementary sets, which also leads to representation formulas for the divergence measure of the extension and yields estimates on the L^∞-norm of the normal traces on $\partial^* U$.

Theorem 5.3. *Let $U \subset\subset \Omega$ be open sets with $\mathscr{H}^{n-1}(\partial U) < \infty$, $F_1 \in \mathcal{DM}^\infty(U;\mathbb{R}^n)$ and $F_2 \in \mathcal{DM}^\infty(\Omega \setminus \overline{U};\mathbb{R}^n)$. Then, if we set*

$$\hat{F}_1(x) := \begin{cases} F_1(x) & \text{if } x \in U, \\ 0 & \text{if } x \in \Omega \setminus U, \end{cases} \quad \text{and} \quad \hat{F}_2(x) := \begin{cases} 0 & \text{if } x \in \overline{U}, \\ F_2(x) & \text{if } x \in \Omega \setminus \overline{U}, \end{cases}$$

we have $\hat{F}_1, \hat{F}_2 \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ with

$$\|\hat{F}_1\|_{\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)} \le \Big(1 + 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}\mathscr{H}^{n-1}(\partial U)\Big)\|F_1\|_{\mathcal{DM}^\infty(U;\mathbb{R}^n)}, \tag{5.6}$$

$$\|\hat{F}_2\|_{\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)} \le \Big(1 + 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}\mathscr{H}^{n-1}(\partial U)\Big)\|F_2\|_{\mathcal{DM}^\infty(\Omega\setminus\overline{U};\mathbb{R}^n)}. \tag{5.7}$$

If we set $F := \hat{F}_1 + \hat{F}_2$, we have $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^n)$ and we obtain the following representation formula for the divergence measure of the extension:

$$\operatorname{div} F = \chi_{U^1} \operatorname{div} \hat{F}_1 + \chi_{U^0} \operatorname{div} \hat{F}_2 + ((\hat{\mathfrak{F}}_{1,i} \cdot \nu_U) - (\hat{\mathfrak{F}}_{2,e} \cdot \nu_U))\mathscr{H}^{n-1} \llcorner \partial^* U, \tag{5.8}$$

where $(\hat{\mathfrak{F}}_{1,i} \cdot \nu_U)$ is the interior normal trace on $\partial^* U$ of $\hat{F}_1$ and $(\hat{\mathfrak{F}}_{2,e} \cdot \nu_U)$ is the exterior normal trace in $\partial^* U$ of $\hat{F}_2$. In particular,

$$\chi_{\partial^* U} \operatorname{div} F = ((\hat{\mathfrak{F}}_{1,i} \cdot \nu_U) - (\hat{\mathfrak{F}}_{2,e} \cdot \nu_U))\mathscr{H}^{n-1} \llcorner \partial^* U. \tag{5.9}$$

In addition, the normal traces of F on $\partial^* U$ satisfy

$$(\mathfrak{F}_i \cdot \nu_U) = (\hat{\mathfrak{F}}_{1,i} \cdot \nu_U), \quad (\mathfrak{F}_e \cdot \nu_U) = (\hat{\mathfrak{F}}_{2,e} \cdot \nu_U) \quad \text{and} \quad (\hat{\mathfrak{F}}_{1,e} \cdot \nu_U) = 0 = (\hat{\mathfrak{F}}_{2,i} \cdot \nu_U)$$

$\mathscr{H}^{n-1}$-a.e. on $\partial^* U$. Finally, we have the following L^∞-estimates of the normal traces:

$$\|\mathfrak{F}_i \cdot \nu_U\|_{L^\infty(\partial^* U;\mathscr{H}^{n-1})} \le \inf_{\varepsilon>0} \|F_1\|_{L^\infty(U_\varepsilon;\mathbb{R}^n)}, \tag{5.10}$$

$$\|\mathfrak{F}_e \cdot \nu_U\|_{L^\infty(\partial^* U;\mathscr{H}^{n-1})} \le \inf_{\varepsilon>0} \|F_2\|_{L^\infty(U^\varepsilon;\mathbb{R}^n)}, \tag{5.11}$$

where $U_\varepsilon := \{x \in U : \operatorname{dist}(x,\partial U) < \varepsilon\}$ and $U^\varepsilon := \{x \in \Omega \setminus \overline{U} : \operatorname{dist}(x,\partial U) < \varepsilon\}$.

Proof. Clearly, $\hat{F}_1, \hat{F}_2$, and F are in $L^\infty(\Omega;\mathbb{R}^n)$. We notice that, since $\mathscr{H}^{n-1}(\partial U) < \infty$, we have $|\partial U| = 0$ and hence we can ignore ∂U when dealing with $\mathscr{L}^n$.

First, we study $\hat{F}_1$. Let U_k be the sequence of approximating sets given in Proposition 5.2. We observe that each U_k is a set of finite perimeter in $\mathbb{R}^n$, since $\mathscr{H}^{n-1}(\partial U_k) < \infty$, and that $U_k \subset\subset U$ implies $U_k^1 \subset \overline{U_k} \subset U$. Hence, for any $\varphi \in C_c^1(\Omega)$ with $\|\varphi\|_\infty \le 1$, we may apply the Gauss–Green formula (4.1):

$$\int_{U_k} F_1 \cdot \nabla\varphi \, dx = -\int_{\partial^* U_k} \varphi(\mathfrak{F}_{i,1} \cdot \nu_{U_k}) \, d\mathscr{H}^{n-1} - \int_{U_k^1} \varphi \, d \operatorname{div} F_1.$$

Thus, by Proposition 5.2,

$$\begin{aligned}\Big|\int_{U_k} F_1 \cdot \nabla\varphi \, dx\Big| &\le |\operatorname{div} F_1|(U_k^1) + \|F_1\|_{L^\infty(U_k;\mathbb{R}^n)}\mathscr{H}^{n-1}(\partial^* U_k) \\ &\le |\operatorname{div} F_1|(U) + \|F_1\|_{L^\infty(U;\mathbb{R}^n)}\mathscr{H}^{n-1}(\partial U_k).\end{aligned}$$

By letting $k \to +\infty$, Lebesgue's dominated convergence theorem and Proposition 5.2 yield

$$\Big|\int_U F_1 \cdot \nabla\varphi \, dx\Big| \le |\operatorname{div} F_1|(U) + 2^{n-1}\frac{n\omega_n}{\omega_{n-1}}\|F_1\|_{L^\infty(U;\mathbb{R}^n)}\mathscr{H}^{n-1}(\partial U).$$

Since we have

$$\int_U F_1 \cdot \nabla\varphi \, dx = \int_\Omega \hat{F}_1 \cdot \nabla\varphi \, dx,$$

it follows that

$$|\operatorname{div} \hat{F}_1|(\Omega) \le |\operatorname{div} F_1|(U) + \|F_1\|_{L^\infty(U;\mathbb{R}^n)} 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \mathscr{H}^{n-1}(\partial U)$$

and so

$$\|\hat{F}_1\|_{\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)} \le |\operatorname{div} F_1|(U) + \|F_1\|_{L^\infty(U;\mathbb{R}^n)} \Big(1 + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \mathscr{H}^{n-1}(\partial U)\Big),$$

which implies (5.6).

Now we consider $\hat{F}_2$ and we take the sequence of open sets W_k in Proposition 5.2. Each W_k is a set of finite perimeter in $\mathbb{R}^n$ since $\mathscr{H}^{n-1}(\partial W_k) < \infty$ and $U \subset\subset W_k$ implies $W_k^0 = (\Omega \setminus W_k)^1 \subset \Omega \setminus W_k \subset \Omega \setminus \overline{U}$. For any $\varphi \in C_c^1(\Omega)$ with $\|\varphi\|_\infty \le 1$, we can apply the integration by parts formula (4.19) to the set $\Omega \setminus W_k$ and the field F_2:

$$\int_{\Omega \setminus W_k} F_2 \cdot \nabla\varphi \, dx = - \int_{\partial^* W_k} \varphi(\mathfrak{F}_{i,2} \cdot \nu_{\Omega\setminus W_k}) \, d\mathscr{H}^{n-1} - \int_{W_k^0} \varphi \, d \operatorname{div} F_2.$$

Letting $k \to +\infty$, we obtain, by Proposition 5.2 and Lebesgue's dominated convergence theorem,

$$\Big| \int_{\Omega \setminus U} F_2 \cdot \nabla\varphi \, dx \Big| \le |\operatorname{div} F_2|(\Omega \setminus \overline{U}) + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \|F_2\|_{L^\infty(\Omega\setminus\overline{U};\mathbb{R}^n)} \mathscr{H}^{n-1}(\partial U).$$

Hence, since we have

$$\int_{\Omega\setminus\overline{U}} F_2 \cdot \nabla\varphi \, dx = \int_\Omega \hat{F}_2 \cdot \nabla\varphi \, dx,$$

taking the sup in φ we obtain

$$|\operatorname{div} \hat{F}_2|(\Omega) \le |\operatorname{div} F_2|(\Omega \setminus \overline{U}) + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \|F_2\|_{L^\infty(\Omega\setminus\overline{U};\mathbb{R}^n)} \mathscr{H}^{n-1}(\partial U),$$

and so

$$\|\hat{F}_2\|_{\mathcal{DM}^\infty(\Omega;\mathbb{R}^n)} \le |\operatorname{div} F_2|(\Omega \setminus \overline{U}) + \Big(1 + 2^{n-1} \frac{n\omega_n}{\omega_{n-1}} \mathscr{H}^{n-1}(\partial U)\Big) \|F_2\|_{L^\infty(\Omega\setminus\overline{U};\mathbb{R}^n)},$$

which implies (5.7). It is then clear that $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$.

As for the second part of the statement, we notice that, for any $\varphi \in C_c^1(\Omega)$, we can apply (4.1) to U and (4.19) to $\Omega \setminus U$, thus obtaining

$$\begin{aligned} \int_\Omega F \cdot \nabla\varphi \, dx &= \int_U \hat{F}_1 \cdot \nabla\varphi \, dx + \int_{\Omega\setminus U} \hat{F}_2 \cdot \nabla\varphi \, dx \\ &= - \int_{U^1} \varphi \, d \operatorname{div} \hat{F}_1 - \int_{\partial^* U} \varphi(\hat{\mathfrak{F}}_{1,i} \cdot \nu_U) \, d\mathscr{H}^{n-1} - \int_{U^0} \varphi \, d \operatorname{div} \hat{F}_2 - \int_{\partial^* U} \varphi(\hat{\mathfrak{F}}_{2,i} \cdot \nu_{\Omega\setminus U}) \, d\mathscr{H}^{n-1}. \end{aligned}$$

By (4.24), we get (5.8) and (5.9). Applying again formulas (4.1) to U and (4.19) to $\Omega \setminus U$, we get

$$\begin{aligned} \int_U F \cdot \nabla\varphi \, dx &= - \int_{U^1} \varphi \, d \operatorname{div} F - \int_{\partial^* U} \varphi(\mathfrak{F}_i \cdot \nu_U) \, d\mathscr{H}^{n-1} \\ &= \int_U \hat{F}_1 \cdot \nabla\varphi \, dx = - \int_{U^1} \varphi \, d \operatorname{div} \hat{F}_1 - \int_{\partial^* U} \varphi(\hat{\mathfrak{F}}_{1,i} \cdot \nu_U) \, d\mathscr{H}^{n-1}, \\ \int_{\Omega\setminus U} F \cdot \nabla\varphi \, dx &= - \int_{U^0} \varphi \, d \operatorname{div} F - \int_{\partial^* U} \varphi(\mathfrak{F}_i \cdot \nu_{\Omega\setminus U}) \, d\mathscr{H}^{n-1} \\ &= \int_{\Omega\setminus U} \hat{F}_2 \cdot \nabla\varphi \, dx = - \int_{U^0} \varphi \, d \operatorname{div} \hat{F}_2 - \int_{\partial^* U} \varphi(\hat{\mathfrak{F}}_{2,i} \cdot \nu_{\Omega\setminus U}) \, d\mathscr{H}^{n-1}, \end{aligned}$$

which, together with (4.24) and (5.8), yields $(\mathfrak{F}_i \cdot \nu_U) = (\hat{\mathfrak{F}}_{1,i} \cdot \nu_U)$, $(\mathfrak{F}_e \cdot \nu_U) = (\hat{\mathfrak{F}}_{2,e} \cdot \nu_U)$ $\mathscr{H}^{n-1}$-a.e. on $\partial^* U$.

Finally, using the estimates in Remark 4.3 to the field F, we obtain (5.10) and (5.11). If we apply these estimates to $\hat{F}_1$ and $\hat{F}_2$, we have $(\hat{\mathfrak{F}}_{1,e} \cdot \nu_U) = 0 = (\hat{\mathfrak{F}}_{2,i} \cdot \nu_U)$, since $\hat{F}_1 = 0$ in U^ε and $\hat{F}_2 = 0$ in U_ε. □

Remark 5.4. It should be noted that the normal traces $(\hat{\mathfrak{F}}_{i,j} \cdot \nu), (\hat{\mathfrak{F}}_{e,j} \cdot \nu)$ are the densities with respect to $\mathscr{H}^{n-1} \llcorner \partial^* U$ of the Radon measures

$$\overline{2\chi_U \hat{F}_j \cdot D\chi_U} \quad \text{and} \quad \overline{2\chi_{\Omega\setminus U} \hat{F}_j \cdot D\chi_U},$$

respectively, for $j = 1, 2$. Then it is clear that

$$\overline{2\chi_U \hat{F}_2 \cdot D\chi_U} = \overline{2\chi_{\Omega\setminus U} \hat{F}_1 \cdot D\chi_U} = 0,$$

since $\hat{F}_2 = 0$ in U and $\hat{F}_1 = 0$ in $\Omega \setminus U$.

In particular, we see that, if the topological and measure theoretic interior and exterior of U coincide up to an $\mathscr{H}^{n-1}$-negligible set, we obtain a representation formula for the divergence measure of the extension in terms of the divergences of the fields as well as new Gauss–Green formulas up to the boundary of the smaller domain U.

Corollary 5.5. *In the hypotheses of Theorem 5.3, if $\mathscr{H}^{n-1}(U^1 \setminus U) = 0$ and $\mathscr{H}^{n-1}(U^0 \setminus (\Omega \setminus \overline{U})) = 0$, or, equivalently, $\mathscr{H}^{n-1}(\partial U \setminus \partial^* U) = 0$, we have*

$$\operatorname{div} F = \chi_U \operatorname{div} F_1 + \chi_{\Omega\setminus\overline{U}} \operatorname{div} F_2 + ((\hat{\mathfrak{F}}_{1,i} \cdot \nu_U) - (\hat{\mathfrak{F}}_{2,e} \cdot \nu_U))\mathscr{H}^{n-1} \llcorner \partial U, \tag{5.12}$$

$$|\operatorname{div} F|(\Omega) \le |\operatorname{div} F_1|(U) + |\operatorname{div} F_2|(\Omega \setminus \overline{U}) + \mathscr{H}^{n-1}(\partial U) \inf_{\varepsilon>0}(\|F_1\|_{L^\infty(U_\varepsilon;\mathbb{R}^n)} + \|F_2\|_{L^\infty(U^\varepsilon;\mathbb{R}^n)}), \tag{5.13}$$

$$\operatorname{div} F_1(U) = -\int_{\partial U} \hat{\mathfrak{F}}_{1,i} \cdot \nu_U \, d\mathscr{H}^{n-1}, \tag{5.14}$$

$$\operatorname{div} F_1(U) = -\operatorname{div} F(\partial U) - \int_{\partial U} \hat{\mathfrak{F}}_{2,e} \cdot \nu_U \, d\mathscr{H}^{n-1}. \tag{5.15}$$

In particular, if $F_2 = 0$, then

$$\operatorname{div} F_1(U) = -\int_{\partial U} \hat{\mathfrak{F}}_{1,i} \cdot \nu_U \, d\mathscr{H}^{n-1} = -\operatorname{div} \hat{F}_1(\partial U). \tag{5.16}$$

Proof. One has the topological and measure theoretic decompositions

$$U \cup \partial U \cup (\Omega \setminus \overline{U}) = \Omega = U^1 \cup U^0 \cup \partial^* U \cup \mathcal{Z}, \quad \mathscr{H}^{n-1}(\mathcal{Z}) = 0.$$

By hypothesis, one also has

$$U^1 = U \cup Z_U, \quad U^0 = (\Omega \setminus \overline{U}) \cup Z_{\Omega\setminus U} \quad \text{with } \mathscr{H}^{n-1}(Z_U) = 0 = \mathscr{H}^{n-1}(Z_{\Omega\setminus U}) = 0,$$

which yields $\partial U = Z_U \cup Z_{\Omega\setminus U} \cup \partial^* U \cup \mathcal{Z}$ and $\mathscr{H}^{n-1}(\partial U \setminus \partial^* U) = 0$. Analogously, if $\mathscr{H}^{n-1}(\partial U \setminus \partial^* U) = 0$, then $\partial U = \partial^* U \cup Z_{\partial U}$ with $\mathscr{H}^{n-1}(Z_{\partial U}) = 0$. Hence one has $U^1 = U \cup (Z_{\partial U} \cap U^1)$ and $U^0 = (\Omega \setminus \overline{U}) \cup (Z_{\partial U} \cap U^0)$, which implies $\mathscr{H}^{n-1}(U^1 \setminus U) = 0$ and $\mathscr{H}^{n-1}(U^0 \setminus (\Omega \setminus \overline{U})) = 0$.

Now, since the divergence of a $\mathcal{DM}^\infty$-field is absolutely continuous with respect to $\mathscr{H}^{n-1}$, we can work with U and $\Omega \setminus \overline{U}$ instead of U^1 and U^0, respectively. It is easy to see that $\operatorname{div} \hat{F}_1 = \operatorname{div} F_1$ in $\mathcal{M}(U)$. Indeed, for any $\varphi \in C_c^1(U)$,

$$\int_U \varphi \, d\operatorname{div} \hat{F}_1 = -\int_U \hat{F}_1 \cdot \nabla\varphi \, dx = -\int_U F_1 \cdot \nabla\varphi \, dx = \int_U \varphi \, d \operatorname{div} F_1.$$

Then, by the density of $C_c^1(U;\mathbb{R}^n)$ in $C_c(U;\mathbb{R}^n)$ with respect to the sup norm, we can conclude the equality of the Radon measures. Analogously, $\operatorname{div} \hat{F}_2 = \operatorname{div} F_2$ in $\mathcal{M}(\Omega \setminus \overline{U})$. Therefore, (5.8) implies (5.12). Estimate (5.13) follows immediately, also by (5.10) and (5.11).

It remains to justify the Gauss–Green formulas (5.14)–(5.16). We begin by observing that (3.11) implies

$$\operatorname{div}(\chi_U F) = \chi_U \operatorname{div} F + 2\overline{\chi_U F \cdot D\chi_U}.$$

We now evaluate over Ω, using the fact that U is bounded and using Lemma 3.1, to find

$$\operatorname{div} F(U) = -\int_{\partial U} (\mathfrak{F}_i \cdot \nu_U) \, d\mathscr{H}^{n-1}.$$

Then, since $\operatorname{div} F = \operatorname{div} F_1$ in $\mathcal{M}(U)$ and $(\mathfrak{F}_i \cdot \nu_U) = (\hat{\mathfrak{F}}_{1,i} \cdot \nu_U)$ $\mathscr{H}^{n-1}$-a.e. on ∂U by Theorem 5.3, one has (5.14).

With a similar argument, we can show that

$$\operatorname{div}(\chi_U F) = \chi_{U\cup\partial U}\operatorname{div} F + 2\overline{\chi_{\Omega\setminus U} F\cdot D\chi_U},$$

and so we obtain

$$\operatorname{div} F(U\cup\partial U) = -\int_{\partial U} (\mathcal{F}_e\cdot\nu_U)\, d\mathcal{H}^{n-1}.$$

Using the fact that $(\mathcal{F}_e\cdot\nu_U) = (\hat{\mathcal{F}}_{2,e}\cdot\nu_U)$ $\mathcal{H}^{n-1}$-a.e. on ∂U by Theorem 5.3, we get (5.15). Finally, if $F_2 = 0$, by Theorem 5.3, we have $(\hat{\mathcal{F}}_{2,e}\cdot\nu_U) = 0$ $\mathcal{H}^{n-1}$-a.e. on ∂U and so (5.14) and (5.15) imply (5.16). □

Remark 5.6. We notice that if U is an open bounded set with Lipschitz boundary, then Remark 2.10 implies $\mathcal{H}^{n-1}(\partial U\setminus\partial^* U) = 0$. Hence, U satisfies the hypotheses of Corollary 5.5. For bounded open sets U satisfying the hypotheses of Corollary 5.5 and for $F\in\mathcal{DM}^\infty(U;\mathbb{R}^n)$, Theorem 5.3 shows that the zero extension $\hat{F}$ belongs to $\mathcal{DM}^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and that there exists the interior normal trace $(\hat{\mathcal{F}}_i\cdot\nu_U)$ on ∂U, while the exterior normal trace is zero. By formulas (5.16) and (5.12) of Corollary 5.5, we have

$$\operatorname{div} F(U) = -\int_{\partial U} (\hat{\mathcal{F}}_i\cdot\nu_U)\, d\mathcal{H}^{n-1} \quad\text{and}\quad \chi_{\partial U}\operatorname{div}\hat{F} = (\hat{\mathcal{F}}_i\cdot\nu_U)\mathcal{H}^{n-1}\llcorner\partial U.$$

In addition, by Theorem 2.18 and (5.12), for any $\varphi\in\operatorname{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ we obtain

$$\operatorname{div}(\varphi\hat{F}) = \varphi\operatorname{div}\hat{F} + \hat{F}\cdot\nabla\varphi = \varphi\chi_U\operatorname{div} F + \varphi(\hat{\mathcal{F}}_i\cdot\nu_U)\mathcal{H}^{n-1}\llcorner\partial U + \chi_U F\cdot\nabla\varphi. \tag{5.17}$$

Hence, since $\varphi\hat{F}$ has compact support in $\mathbb{R}^n$, we can evaluate (5.17) on $\mathbb{R}^n$ and apply Lemma 3.1 to obtain the following integration by parts formula:

$$\int_U \varphi\, d\operatorname{div} F = -\int_{\partial U}\varphi(\hat{\mathcal{F}}_i\cdot\nu_U)\, d\mathcal{H}^{n-1} - \int_U F\cdot\nabla\varphi\, dx.$$

6 Concluding remarks

In this section, we would like to make some final remarks concerning the results we have obtained and comparisons with other related results in the literature. First, we briefly discuss the importance of choosing $p=\infty$ in the question of the existence of normal traces for $F\in\mathcal{DM}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$. Then we indicate some relations between our $p=\infty$ theory and known alternate approaches, which will lead to some known variants of what we have presented. In particular, we will illustrate how one can obtain the consistency of the normal traces with the classical dot product $F\cdot\nu_E$ without the assumption that F is continuous (as made in Theorem 3.7) provided that one makes additional assumptions on F and E. We will also discuss alternate representations of the normal trace as certain local averages.

We begin by illustrating why $F\in\mathcal{DM}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for $p<\infty$ may fail to admit locally integrable interior and exterior normal traces which satisfy the Gauss–Green formula. The example relies heavily on a construction of Šilhavý in his study of $\mathcal{DM}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$-fields (see [25, Example 3.3 and Proposition 6.1]).

Example 6.1. For any $n\geq 2$ and any $p\in[1,\infty)$ there exists a vector field $F\in\mathcal{DM}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\setminus\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ for which we can find a set $E\subset\subset\Omega$ of finite perimeter in Ω such that there do not exist interior and exterior normal traces $(\mathcal{F}_i\cdot\nu_E), (\mathcal{F}_e\cdot\nu_E)\in L^1(\partial^*E;\mathcal{H}^{n-1})$ satisfying respectively

$$\int_{E^1}\varphi\, d\operatorname{div} F = -\int_{\partial^* E}\varphi\mathcal{F}_i\cdot\nu_E\, d\mathcal{H}^{n-1} - \int_E \nabla\varphi\cdot F\, dx \tag{6.1}$$

and

$$\int_{E^1\cup\partial^* E}\varphi\, d\operatorname{div} F = -\int_{\partial^* E}\varphi\mathcal{F}_e\cdot\nu_E\, d\mathcal{H}^{n-1} - \int_E \nabla\varphi\cdot F\, dx \tag{6.2}$$

for any $\varphi\in C_c^\infty(\Omega)$.

Indeed, as in [25], we will make use of the vector field F which is the gradient of a Newtonian potential of uniform mass distribution on a suitable compact set K of Hausdorff dimension $m \in (0, n-1)$. Without loss of generality, we may assume that $B(0,1) \subset\subset \Omega$. For any $m \in (0, n-1)$ we choose a compact set $K \subset B(0,1) \cap \{x \in \mathbb{R}^n : x_n = 0\}$ with $0 < \mathscr{H}^m(K) < \infty$ for which there is a constant $c > 0$ such that

$$\mathscr{H}^m(K \cap B(x,r)) \le cr^m \quad \text{for all } x \in \mathbb{R}^n \text{ and all } r > 0.$$

For the existence of such a set K, see [15, Corollary 4.12]. We define the vector field $\mathcal{L}^n$-a.e. on Ω by the formula

$$F(x) := \frac{1}{n\omega_n} \int_K \frac{(x-y)}{|x-y|^n} \, d\mathscr{H}^m(y).$$

Following the calculations of [25, Proposition 6.1], one sees that $F \in L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ provided that

$$m > n - \frac{p}{p-1} \tag{6.3}$$

and that in such cases $F \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ with

$$\operatorname{div} F = \mathscr{H}^m \llcorner K \quad \text{in } \mathcal{M}(\Omega),$$

and hence $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ provided that $m \in (0, n-1)$ can be chosen to satisfy (6.3) with n and p given. For $p \in [1, \frac{n}{n-1}]$ any $m \in (0, n-1)$ will do, while for $\frac{n}{n-1} < p < \infty$ one can choose such an m since $n - \frac{p}{p-1} \in (0, n-1)$. In addition, $F \notin \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ since $\operatorname{div} F$ is not $\mathscr{H}^{n-1}$-absolutely continuous, as follows from Corollary 2.16.

With $E := B(0,1) \cap \{x_n > 0\}$, we claim that the existence of normal traces satisfying (6.1)–(6.2) leads to a contradiction. To this end, we note that for any Borel set $A \subset \Omega$ one has

$$\operatorname{div} F(A) = \operatorname{div} F(A \cap K) = \operatorname{div} F(A \cap \{x_n = 0\}).$$

Subtracting (6.1) from (6.2) would say that for each $\varphi \in C^\infty_c(\Omega)$ one has

$$\int_{\partial^*(B(0,1)\cap\{x_n>0\})} \varphi \, d \operatorname{div} F = - \int_{\partial^*(B(0,1)\cap\{x_n>0\})} \varphi(\mathcal{F}_e \cdot \nu_E - \mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1}. \tag{6.4}$$

We observe that

$$\partial^*(B(0,1) \cap \{x_n > 0\}) = (B(0,1) \cap \{x_n = 0\}) \cup (\partial B(0,1) \cap \{x_n > 0\}).$$

Since $\mathscr{H}^m(K) < \infty$, one has $\operatorname{cap}_{n-m}(K) \le \operatorname{cap}_{n-m}(K, \Omega) = 0$ by Theorem 2.6 and hence, by Lemma 2.8, there exists a sequence $\varphi_j \in C^\infty_c(\Omega)$ which satisfies $0 \le \varphi_j \le 1$, $\varphi_j = 1$ on K and $\varphi_j(x) \to 0$ for all $x \in \Omega \setminus K$. We can write equation (6.4) for any φ_j and, since the measure $\operatorname{div} F$ is supported in K, we have

$$\int_{\partial^*(B(0,1)\cap\{x_n>0\})} \varphi_j \, d \operatorname{div} F = \int_K d \operatorname{div} F = \mathscr{H}^m(K) > 0.$$

On the other hand, $\varphi_j \to 0$ $\mathscr{H}^{n-1}$-a.e. since Theorem 2.6 shows that $\operatorname{cap}_{n-m}(K) = 0$ implies $\mathscr{H}^s(K) = 0$ for any $s > m$, hence in particular for $s = n-1$. Thus we may apply Lebesgue's dominated convergence theorem to the right-hand side of (6.4), since $0 \le \varphi_j \le 1$ and $(\mathcal{F}_e \cdot \nu_E - \mathcal{F}_i \cdot \nu_E) \in L^1(\partial^*(B(0,1) \cap \{x_n > 0\}); \mathscr{H}^{n-1})$. In this way, we obtain

$$\mathscr{H}^m(K) = \lim_{j\to+\infty} \int_{\partial^*(B(0,1)\cap\{x_n>0\})} \varphi_j \, d \operatorname{div} F = \lim_{j\to+\infty} - \int_{\partial^*(B(0,1)\cap\{x_n>0\})} \varphi_j(\mathcal{F}_e \cdot \nu_E - \mathcal{F}_i \cdot \nu_E) \, d\mathscr{H}^{n-1} = 0,$$

which contradicts the positivity of $\mathscr{H}^m(K)$.

It is interesting to notice that the obstruction to the existence of normal traces which complete Gauss–Green formulas such as (6.1)–(6.2) is the possibility of having $\operatorname{div} F$ supported on a set of Hausdorff dimension strictly less than $n-1$ which lies on the reduced boundary of a set of finite perimeter. However, one knows that it is possible to recover such formulas also in the case $F \in \mathcal{DM}^p_{\text{loc}}(\Omega;\mathbb{R}^n) \setminus \mathcal{DM}^\infty_{\text{loc}}(\Omega;\mathbb{R}^n)$, provided that F and the set E of finite perimeter in Ω satisfy some additional assumptions. We refer to Degiovanni, Marzocchi and Musesti [13], Schuricht [24] and Šilhavý [25] for a complete treatment of this theory. Here we only discuss how their results are consistent with ours in the case $p = \infty$.

We begin with the question of the consistency of normal traces with the classical dot product even when F is not continuous, provided that F and E satisfy two additional conditions which were in introduced in [13] and exploited in greater generality in [24]. These conditions are

$$|\operatorname{div} F|(\partial^* E) = 0 \quad \text{and} \quad \int_{\partial^* E} h \, d\mathscr{H}^{n-1} < \infty, \tag{6.5}$$

where $h \in L^1_{\text{loc}}(\Omega)$ is a nonnegative function such that one can extract a subsequence $\{F_k\}_{k\in\mathbb{N}}$ of the canonical mollification $F_k := F * \rho_{\varepsilon_k}$ of $F \in L^1_{\text{loc}}(\Omega;\mathbb{R}^n)$ satisfying[3]

$$F_k \to F \quad \text{in } L^1_{\text{loc}}(\Omega;\mathbb{R}^n), \tag{6.6}$$

$$F_k(x) \to F(x) \quad \text{for each } x \in \Omega \text{ such that } h(x) < +\infty, \tag{6.7}$$

$$|F_k(x)| \leq h(x) \quad \text{for each } x \in \Omega \text{ and } k \in \mathbb{N}. \tag{6.8}$$

The existence of such a nonnegative function h for which the above properties hold is standard (see, for example, [4, Theorem 4.9]). For $F \in \mathcal{DM}^1_{\text{loc}}(\Omega;\mathbb{R}^n)$ and $E \subset \Omega$ of finite perimeter in Ω satisfying conditions (6.5), [24, Proposition 5.11] gives the integration by parts formula[4]

$$\int_{E^1} \varphi \, d \operatorname{div} F = -\int_{\partial^* E} \varphi F \cdot \nu_E \, d\mathscr{H}^{n-1} - \int_E F \cdot \nabla\varphi \, dx, \tag{6.9}$$

for every $\varphi \in \operatorname{Lip}_{\text{loc}}(\Omega)$ such that $\chi_E \varphi$ has compact support in Ω.

Remark 6.2. For $F \in \mathcal{DM}^\infty_{\text{loc}}(\Omega;\mathbb{R}^n)$ and $E \subset \Omega$ of finite perimeter in Ω satisfying conditions (6.5), the interior and exterior normal traces of F on $\partial^* E$ coincide $\mathscr{H}^{n-1}$-a.e. on $\partial^* E$ with the classical dot product $F \cdot \nu_E$ and one has the formula (6.9) for each $\varphi \in \operatorname{Lip}_{\text{loc}}(\Omega)$ such that $\chi_E \varphi$ has compact support in Ω. Indeed, since $|\operatorname{div} F|(\partial^* E) = 0$ by the first condition in (6.5), the interior and exterior normal traces of F coincide and so $\overline{\chi_E F \cdot D\chi_E} = \overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}$ in $\mathcal{M}_{\text{loc}}(\Omega)$. Thus one has the following identities in $\mathcal{M}_{\text{loc}}(\Omega)$:

$$\overline{F \cdot D\chi_E} = \overline{\chi_E F \cdot D\chi_E} + \overline{\chi_{\Omega\setminus E} F \cdot D\chi_E} = 2\overline{\chi_E F \cdot D\chi_E} = 2\overline{\chi_{\Omega\setminus E} F \cdot D\chi_E}. \tag{6.10}$$

For any $\varphi \in C^1_c(\Omega)$ one has

$$\lim_{k\to+\infty} \int_\Omega \varphi F_k \cdot dD\chi_E = \lim_{k\to+\infty} \int_{\partial^* E} \varphi F_k \cdot \nu_E \, d\mathscr{H}^{n-1} = \int_{\partial^* E} \varphi F \cdot \nu_E \, d\mathscr{H}^{n-1}, \tag{6.11}$$

by Lebesgue's dominated convergence theorem with respect to the measure $|D\chi_E| = \mathscr{H}^{n-1} \llcorner \partial^* E$, since $F_k(x) \to F(x)$ for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$ and $|\varphi F_k \nu_E| \leq Ch$ which is summable on $\partial^* E$ by the second condition in (6.5). In addition, since $\varphi F_k \in C^1_c(\Omega;\mathbb{R}^n)$, one has

$$\begin{aligned} \int_\Omega \varphi F_k \cdot dD\chi_E &= -\int_\Omega \chi_E \operatorname{div}(\varphi F_k) \, dx \\ &= -\int_\Omega \chi_E \varphi \operatorname{div}(F_k) \, dx - \int_\Omega \chi_E \nabla\varphi \cdot F_k \, dx \\ &= -\int_\Omega \int_E \varphi(x) \rho_{\varepsilon_k}(x-y) \, dx \, d\operatorname{div} F(y) - \int_\Omega \chi_E \nabla\varphi \cdot F_k \, dx. \end{aligned}$$

3 Here and below we will still denote by F the particular representative which is the limit of the sequence F_k in the sense (6.7).
4 Schuricht actually treats divergence tensor fields $F \in \mathcal{DM}^1_{\text{loc}}(\Omega;\mathbb{R}^{n\times m})$ and uses the opposite orientation with respect to our choice. See also the related Theorems 5.2 and 5.4 in [13].

Now, notice that for $y \in E^1$ one has

$$\int_E \varphi(x)\rho_{\varepsilon_k}(x-y)\,dx \to \varphi(y),$$

while if $y \in E^0$, one has

$$\int_E \varphi(x)\rho_{\varepsilon_k}(x-y)\,dx \to 0.$$

On the other hand, since $\mathscr{H}^{n-1}(\partial^m E \setminus \partial^* E) = 0$ and $|\operatorname{div} F| \ll \mathscr{H}^{n-1}$, Corollary 2.16 yields

$$|\operatorname{div} F|(\Omega \setminus (E^1 \cup E^0)) = |\operatorname{div} F|(\partial^* E) = 0.$$

Thus, by Lebesgue's dominated convergence theorem and the Leibniz rule (Theorem 2.18), one obtains

$$\begin{aligned}\lim_{k\to+\infty} \int_\Omega \varphi F_k \cdot dD\chi_E &= -\int_\Omega \chi_{E^1}\varphi\, d\operatorname{div} F - \int_\Omega \chi_E \nabla\varphi \cdot F\,dx \\ &= -\int_\Omega \chi_{E^1}\varphi\, d\operatorname{div} F + \int_\Omega \varphi\, d\operatorname{div}(\chi_E F) \\ &= -\int_\Omega \chi_{E^1}\varphi\, d\operatorname{div} F + \int_\Omega \varphi\chi_E^*\, d\operatorname{div} F + \int_\Omega \varphi\, d\overline{F \cdot D\chi_E} \\ &= \int_\Omega \varphi\, d\overline{F \cdot D\chi_E},\end{aligned}$$

since $\chi_E^* = \chi_{E^1}$ on E^1 by (2.12) and $|\operatorname{div} F|(\partial^* E) = 0$. From (6.11) and the density of $C_c^1(\Omega)$ in $C_c(\Omega)$ it follows that $\overline{F \cdot D\chi_E} = F \cdot \nu_E\, d\mathscr{H}^{n-1} \llcorner \partial^* E$, which means, by (6.10), $\mathcal{F}_i \cdot \nu_E = \mathcal{F}_e \cdot \nu_E = F \cdot \nu_E$ $\mathscr{H}^{n-1}$-a.e. on $\partial^* E$.

It is also perhaps worth mentioning that in this setting of $F \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and $E \subset \Omega$ satisfying (6.5), [24, Proposition 6.5] provides the following Leibniz formula for χ_E and the particular representation of F described in (6.7):

$$\operatorname{div}(\chi_E F) = g_E \operatorname{div} F + F \cdot \nu_E \mathscr{H}^{n-1} \llcorner \partial^* E,$$

where $g_E \in L^\infty(\Omega; |\operatorname{div} F|)$ satisfies $0 \le g_E \le 1$ and $g_E(x) = d(E, x)$ at each x for which the Lebesgue density (2.6) exists. This last property indicates that g_E is in some sense a generalization of χ_E^*. If one also assumes that $E \subset\subset \Omega$, then, by Lemma 3.1, one has the following Gauss–Green formula for $F \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$:

$$\int_\Omega g_E\, d\operatorname{div} F = -\int_{\partial^* E} F \cdot \nu_E\, d\mathscr{H}^{n-1}.$$

If $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^n)$ and $|\operatorname{div} F|(\partial^* E) = 0$, this formula gives a natural variant to (3.19), since $g_E = \chi_E^*$ $\mathscr{H}^{n-1}$-a.e.

Next we turn our attention to alternate representations of the normal trace functional (4.21) and of the normal traces discussed herein.

Remark 6.3. If $F \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$, then for any compact set $K \subset \Omega$ one can represent the normal trace functional as an average on one-sided tubular neighborhoods of ∂K in the sense that

$$\operatorname{div} F(K) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{K_\varepsilon \setminus K} F \cdot \nu_K^d\, dx, \tag{6.12}$$

where $K_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, K) \le \varepsilon\}$ and $\nu_K^d(x) = \nabla \operatorname{dist}(x, K)$ is a unit vector for $\mathscr{L}^n$-a.e. $x \in \Omega \setminus K$. This last property says that ν_K^d is a sort of generalization of the exterior normal. Indeed, formula (6.12) holds even for $F \in \mathcal{DM}^1_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ (see [24, Theorem 5.20]). It is sufficient to apply the definition of distributional divergence using as tests the Lipschitz functions

$$\varphi_K^\varepsilon(x) := \begin{cases} 1 & \text{if } x \in K, \\ 1 - \frac{1}{\varepsilon}\operatorname{dist}(x, K) & \text{if } x \in K_\varepsilon \setminus K, \\ 0 & \text{if } x \notin K_\varepsilon, \end{cases}$$

which clearly have compact support for ε small enough, and then to pass to the limit as $\varepsilon \to 0$.

Next we notice that it is possible to provide an alternate representation formula for interior and exterior normal traces of F as limits of fluxes in terms of the blow-up construction of De Giorgi (as recalled in the discussion leading to Proposition 4.10). This observation comes from the paper of Šilhavý [25], in which one finds a rich study of the normal trace functional under various summability assumptions on F and concentration hypotheses on $|\operatorname{div} F|$. In particular, we refer to [25, Theorems 4.2, 4.4 and 4.6]. We will comment only on the case $p = \infty$ as treated in [25, Theorem 4.4], where we note that the author treats explicitly only the case of the interior normal trace and uses an orientation which is opposite to ours.

Remark 6.4. Let $F \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ and let $E \subset \Omega$ be a set of locally finite perimeter. Then one has the following formulas for interior and exterior traces which are valid for $\mathscr{H}^{n-1}$-a.e. $x \in \partial^* E$:

$$(\mathcal{F}_i \cdot \nu_E)(x) = \lim_{r \to 0} \frac{n}{\omega_{n-1} r^n} \int\limits_{\Pi^+_{\nu_E}(x) \cap B(x,r)} F(y) \cdot \frac{y - x}{|y - x|}\, dy \tag{6.13}$$

and

$$(\mathcal{F}_e \cdot \nu_E)(x) = -(\mathcal{F}_i \cdot \nu_{\Omega \setminus E})(x) = -\lim_{r \to 0} \frac{n}{\omega_{n-1} r^n} \int\limits_{\Pi^-_{\nu_E}(x) \cap B(x,r)} F(y) \cdot \frac{y - x}{|y - x|}\, dy. \tag{6.14}$$

Indeed, (6.14) follows from (4.24) and (6.13), since $\Pi^+_{\nu_{\Omega \setminus E}}(x) = \Pi^-_{\nu_E}(x)$. In order to establish (6.13), one applies Theorem 4.2 and (4.1) to the Lipschitz function $\varphi_{x,r}(y) := \max\{r - |y - x|, 0\}$, where $x \in \partial^* E$ and $r > 0$. For the details, one can consult [25, Theorem 4.4]; roughly speaking, one needs to exploit the tangential properties of the sets of finite perimeter as in the proof of Proposition 4.10.

We conclude with an application of these formulas to a classical example.

Example 6.5. Consider the field

$$F(x_1, x_2) = \sin\Big(\frac{1}{x_1 - x_2}\Big)(1, 1) \in \mathcal{DM}^{\infty}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2).$$

It is easy to see that $\operatorname{div} F = 0$ in the sense of distributions, hence the interior and exterior normal traces of F always coincide by (4.7). We are interested in finding the normal trace on the line $\{x_1 = x_2\}$; that is, on the set of essential singularities, in any neighborhood of which F is not even a function of bounded variation. Hence, let $x = (t, t)$, $\nu = \frac{\sqrt{2}}{2}(1, -1)$ and $E = \Pi^+_\nu(x)$. By a roto-translation and a passage to polar coordinates, we have

$$\begin{aligned}
\int\limits_{\Pi^+_\nu(x) \cap B(x,r)} \sin\Big(\frac{1}{y_1 - y_2}\Big) \frac{y_1 - x_1 + y_2 - x_2}{|y - x|}\, dy &= \int\limits_{\Pi^+_\nu(0) \cap B(0,r)} \sin\Big(\frac{1}{y_1 - y_2}\Big) \frac{y_1 + y_2}{\sqrt{y_1^2 + y_2^2}}\, dy \\
&= \int\limits_{\{z_1 \geq 0\} \cap B(0,r)} \sin\Big(\frac{1}{\sqrt{2} z_1}\Big) \frac{\sqrt{2} z_2}{\sqrt{z_1^2 + z_2^2}}\, dz \\
&= \int_0^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\Big(\frac{1}{\sqrt{2}\rho \cos\theta}\Big) \sqrt{2}\rho \sin\theta \, d\theta\, d\rho = 0,
\end{aligned}$$

since $\sin(\frac{1}{\sqrt{2}\rho\cos\theta}) \sin\theta$ is odd in $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for any $\rho > 0$. Hence, we conclude that

$$(\mathcal{F}_i \cdot \nu_E)(x) = (\mathcal{F}_e \cdot \nu_E)(x) = 0$$

for any $x \in \{x_1 = x_2\}$, by Proposition 4.10. It is possible to prove this identity also using the definition of the normal traces as densities of the Radon measures $2\overline{\chi_E F \cdot D\chi_E}$ and $2\overline{\chi_{\Omega \setminus E} F \cdot D\chi_E}$; however, the method is less straightforward.

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